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The combinatorics of directed planar trees

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A B S T R A C T

We give a geometric realization of the polyhedra governed by the structure of associative algebras with co-inner products, or more precisely, governed by directed planar trees. Our explicit realization of these polyhedra, which include the associahedra in a special case, shows in particular that these polyhedra are homeomorphic to balls. We also calculate the number of vertices of the lowest generalized associahedra, giving appropriate generalizations of the Catalan numbers.

Contents

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1. Introduction

The associahedron, or Stasheff polytope, is a convex polytope whose cellular structure is determined by the combinatorics of planar, rooted trees. In [8,9] Jim Stasheff used these polytopes to study H-spaces up to homotopy, and in particular gave a geometric realization of the associahedron inside a cube. The associahedra appear in many settings in mathematics due to their fundamental definition, and here we note one instance which is relevant for our purposes, namely, the fact that their cellular chains may be used to define the operad of A_∞-algebras (see [7]), giving a resolution of the associative operad.

In this note, we describe a variation of these polytopes, which originally grew out of an attempt to model algebraically string topology operations as defined by Moira Chas and Dennis Sullivan in [2]. In fact, to do so, an essential ingredient consists of a model for the Poincaré duality structure of the underlying space. For example, in [10], the Poincaré duality structure was modeled via a non-degenerate, invariant inner-product with higher homotopies (which were called homotopy inner products). More generally, if one considers an invariant *co*-inner product (with higher homotopies), one may drop the non-degeneracy condition, and still obtain string topology-like operations; this was defined in an algebraic setting in [11]. In this setup one requires n-to-m-operations (i.e. maps $A^{\otimes n} \to A^{\otimes m}$) for each corolla having a *cyclic* order on its inputs and outputs (satisfying the usual edge expansion conditions). Such an algebraic structure on a space A was called a V_∞ algebra in [11]. It is our aim with this paper and two follow-up papers to clarify the combinatorics of this structure as well as identify operadic underpinnings of V_∞ algebras, and furthermore identify the induced space of string topology operations with other models of this space of operations.

In this paper, we take a first step toward analyzing the structure of V_∞ algebras. Using the combinatorics of directed planar trees with a cyclic order α on their exterior vertices (Definition 2.1), we define a cell complex Z_α, our generalization of the associahedron, whose cells are indexed by precisely those trees. This is done using and adding onto the well-known secondary polytope construction of the associahedron defined by Gelfand, Kapranov, and Zelevinsky (see e.g. [4]). We show in Section 3, that Z_α is homeomorphic to a disk, or more precisely, we show the following.

Theorem 3.10. *The space Z_α has the structure of a cell complex where the cells are given by the subspaces Z_T for T in $\mathcal{T}_\alpha$. This structure is a cellular subdivision of the product of an associahedron and a simplex $K_{n_\alpha-1} \times \Delta^{k_\alpha-1}$ in $\mathbb{R}^{n_\alpha} \times \mathbb{R}^{k_\alpha}$, each with their own natural cell complex structures.*

In the case where there are exactly two outgoing edges, and ℓ and m incoming edges (between the two outgoing edges) these polyhedra are precisely the pairahedra as defined in [10]; see Example 3.4(2) below.

In Section 4 we investigate some of the combinatorics of Z_α by studying the number $C(\alpha)$ of vertices of Z_α. We give a recursive formula for calculating $C(\alpha)$ in