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One-factorisations of complete graphs arising from ovals in finite planes[☆]

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ABSTRACT

In this paper we use the geometry of finite planes to set up a procedure for the construction of one-factorisations of the complete graph. Let π be a projective plane of order $n-1$ with n even containing an oval Ω, and regard Ω as the vertex set of the complete graph K_n. Then any one-factorisation of K_n has a representation by a partition of the external points to Ω whose components are of size $\frac{n}{2}$ and meet every tangent to Ω in a unique point. Our goal is to construct such partitions from nice geometric configurations in the Desarguesian plane of order q with $q = p^h$ and $p > 2$ prime.

☆ Dedicated to Professor Alexander Rosa on the occasion of his 80th birthday.

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1. Introduction

A one-factorisation of the complete graph K_n on an even n-vertex set is a partition of the edge set into $n-1$ one-factors, each consisting of $\frac{n}{2}$ edges, that partition the vertex set. Such one-factorisations are frequently required in practical applications, especially in scheduling sporting tournaments where a round robin tournament for teams is to be played in the minimum number of sessions. Their connections to Steiner triple systems, and more generally to Design theory, are also relevant; see for instance [34].

The systematic study of one-factorisations of K_n, initiated in the Seventies by the pioneering work of Rosa, Mendelsohn and Wallis, has been developed around the following three problems; see for instance [8,11,15,16,25,26,32–34] and the references therein.

(I) Enumeration by computer aided exhaustive search for smaller n.
(II) Explicit constructions of infinite families with special features.
(III) Classification and characterisation by automorphism groups and other invariants.

In almost all papers on Problem (II), the "starter-construction" is used, which was introduced by Hartman and Rosa in their seminal paper [12] and generalised later by Buratti [5], so that the methods are fine counting arguments combined with elementary Number theory; see also [3,5,7,9,21,22,27,28,34].

In the more recent paper [14] the "switching method" is introduced, and when this method produces new one-factorisations from a given one is discussed.

In this paper we use a different approach based on the geometry of finite planes. It should be noted that geometric methods in the study of factorisations of graphs or multigraphs are a bit unusual although some results on one-factorisations of multigraphs arising from geometry are found in the literature; see [2,17,18,23,31].

Our basic idea is to represent one-factors of K_n on an oval Ω in a finite projective plane of odd order $n-1$. Since $|\Omega| = n$, we may think about Ω as the vertex-set of K_n so that edges are the chords of Ω. Every chord is uniquely represented by an external point to Ω. In fact, as it happens in the real plane, external points are the common point of two tangents. This shows that each one-factor of K_n defines a set of $\frac{n}{2}$ external points satisfying the "tangent property": no tangent to Ω meets the set in more than one point. Furthermore, $n-1$ of such point-sets of size $\frac{n}{2}$ represent a one-factorisation of K_n if and only if they partition the set of external points to Ω. If this is the case then every tangent meets each of the $n-1$ point-sets in exactly one point.

Although there exist so many one-factorisations, those giving rise to nice geometric configurations seem to be somewhat rare.

In this paper we work out the case where the partition arising from a one-factorisation involving only lines and ovals. For this purpose, the concept of a one-factor represented on Ω by a line or an oval is introduced, and several related preliminary results and open problems are stated; see Section 2. These problems are of independent interest, especially