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A tale of stars and cliques

Tomasz Łuczak [a,1], Joanna Polcyn [a], Christian Reiher [b]

[a] *Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Poznań, Poland*
[b] *Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany*

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ABSTRACT

We show that for infinitely many natural numbers k there are k-uniform hypergraphs which admit a 'rescaling phenomenon' as described in [10]. More precisely, let $\mathcal{A}(k, I, n)$ denote the class of k-graphs on n vertices in which the sizes of all pairwise intersections of edges belong to a set I. We show that if $k = rt^2$ for some $r \geqslant 1$ and $t \geqslant 2$, and I is chosen in some special way, the densest graphs in $\mathcal{A}(rt^2, I, n)$ are either dominated by stars of large degree, or basically, they are 't-thick' rt^2-graphs in which vertices are partitioned into groups of t vertices each and every edge is a union of tr such groups. It is easy to see that, unlike in stars, the maximum degree of t-thick graphs is of a lower order than the number of its edges. Thus, if we study the graphs from $\mathcal{A}(rt^2, I, n)$ with a prescribed number of edges m which minimise the maximum degree, around the value of m which is the number of edges of the largest t-thick graph, a rapid, discontinuous phase transition can be observed. Interestingly, these two types of k-graphs determine the structure of all hypergraphs in $\mathcal{A}(rt^2, I, n)$. Namely, we show that each such hypergraph can be decomposed into a t-thick graph H_T, a special collection H_S of stars, and a sparse 'left-over' graph H_R.

E-mail addresses: tomasz@amu.edu.pl (T. Łuczak), joaska@amu.edu.pl (J. Polcyn), Christian.Reiher@uni-hamburg.de (Chr. Reiher).

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1. Introduction

By a *set system* we mean a pair $S = (V, \mathcal{E})$ such that $\mathcal{E}$ is a collection of subsets of V. The members of V are usually referred to as the *vertices* of the set system, whilst the members of $\mathcal{E}$ are called *edges*. If all members of $\mathcal{E}$ are of the same cardinality $k \geqslant 0$ we call S a *k-uniform hypergraph* or, more briefly, a *k-graph*.

Occasionally we identify a hypergraph H with its set of edges, denoting, for example, by $|H|$ the number of edges in H. For a given set I of nonnegative integers, we say that a k-graph H is *I-intersecting* if $|e \cap f| \in I$ holds for all $e, f \in H$. Starting with the seminal work [5] of Erdős, Ko, and Rado, the study of I-intersecting hypergraphs and set systems has a long tradition in extremal combinatorics (see, e.g., [1,4,2,12,7,8,11] for some milestones). Let us remark that sometimes in the literature (e.g., [4,2,7]) an I-intersecting k-graph on n vertices is called an (n, k, I)-system.

Motivated by the stability of extremal hypergraphs for the 3-uniform loose path of length 3 the first two authors studied $\{0, 2, 3, 4\}$-intersecting 4-graphs in [10]. The aim of the present article is to extend their results to the more general family $\mathcal{J}(r, t)$ which consists of all I-intersecting rt^2-graphs, where $r \geqslant 1$ and $t \geqslant 2$ are arbitrary integers and

$$I = \big\{s\colon t \mid s \text{ or } s \geqslant rt(t-1)\big\}\,.$$

This choice of the set of permissible intersections may look bizarre at first and our main incentive to study it came from the aesthetical merits of the results we hoped to obtain: to explain those, we start from the observation that there are two quite different examples of dense rt^2-graphs $H \in \mathcal{J}(r, t)$ on n vertices with $\Theta(n^{rt})$ edges.

The most obvious one is the *full* $\big(rt(t-1)\big)$*-star*, i.e., a hypergraph H with a distinguished $rt(t-1)$-set S of vertices, called the *centre* of the star, such that the edges of H are precisely the rt^2-supersets of S. Clearly such a star has exactly $\binom{n-rt^2+rt}{rt}$ edges and it can be shown that, for large n, it is the unique hypergraph which maximises the number of edges among all hypergraphs in $\mathcal{J}(r, t)$ on n vertices (see Proposition 2.3 below).

However, there exists another natural construction of dense rt^2-graphs $H \in \mathcal{J}(r, t)$ with n vertices and $\Theta(n^{rt})$ edges. It proceeds by splitting the vertex set into $\lfloor n/t \rfloor$ subsets of size t called *teams* (and a small number of left-over vertices) and to declare an rt^2-set to be an edge if and only if it is a union of rt teams. We call the resulting hypergraph a *thick clique* and to its subhypergraphs we refer as *thick hypergraphs*. Note that each thick hypergraph has the property that for any two edges e and f the number $|e \cap f|$ is a multiple of t and, hence, it indeed belongs to $\mathcal{J}(r, t)$.

The point that interests us here is that even though both the star and the thick clique have $\Theta(n^{rt})$ edges, their maximum vertex degrees are of different orders of magnitude. In fact, while the vertices belonging to the centre of a star have degree $\Omega(n^{rt})$, the maximum degree of a thick clique is easily seen to be only $O(n^{rt-1})$. Perhaps surprisingly, it turns out that this phenomenon arises in a very "discontinuous" manner: As soon as a graph