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Incidence geometry and universality in the tropical plane

Milo Brandt, Michelle Jones, Catherine Lee, Dhruv Ranganathan

A R T I C L E   I N F O

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A B S T R A C T

We examine the incidence geometry of lines in the tropical plane. We prove tropical analogs of the Sylvester–Gallai and Motzkin–Rabin theorems in classical incidence geometry. This study leads naturally to a discussion of the realizability of incidence data of tropical lines. Drawing inspiration from the von Staudt constructions and Mnëv's universality theorem, we prove that determining whether a given tropical linear incidence datum is realizable by a tropical line arrangement requires solving an arbitrary linear programming problem over the integers.

1. Introduction

This paper investigates the incidence geometry of lines and points in the standard tropical plane, taking inspiration from fundamental theorems in combinatorial geometry. Our first results establish tropical versions of two classical theorems in the incidence geometry of $\mathbb{R}^2$: the Sylvester–Gallai theorem and the Motzkin–Rabin theorem.

A set of points $\mathcal{P} \subset \mathbb{R}^2$ in the tropical plane **determines an ordinary tropical line** if there exists a tropical line in $\mathbb{R}^2$ passing through exactly two points in $\mathcal{P}$.

E-mail address: dhruvr@mit.edu (D. Ranganathan).

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Theorem A *(Tropical Sylvester–Gallai). Any set of four or more points in the tropical plane determines at least one ordinary tropical line.*

A set of points $\mathcal{P} \subset \mathbb{R}^2$ with each point coloured either red or blue **determines a monochromatic tropical line** if there exists a tropical line in $\mathbb{R}^2$ containing at least two points of $\mathcal{P}$ and only containing points of the same colour.

Theorem B *(Tropical Motzkin–Rabin). Let $\mathcal{P}$ be a set of four or more points, each coloured red or blue, that determines finitely many tropical lines. Then $\mathcal{P}$ determines at least one monochromatic tropical line.*

We note that the conclusion of Theorem B fails without the supposition that the collection of points $\mathcal{P}$ determines finitely many lines. For instance, four points on the x-axis with alternating colours do not determine a monochromatic tropical line.

We prove the results above by analyzing the Newton subdivision of a dilated simplex that is induced the arrangement of tropical lines projectively dual to $\mathcal{P}$. Classically, a natural successor to the Sylvester–Gallai problem is the determination of the minimum number of ordinary Euclidean lines in an arrangement of given size. Our attempts to address this via Newton subdivisions leads to the following question: *when is a polyhedral subdivision of a dilated standard simplex realized by an arrangement of tropical lines?* One might view this question as being a version of determining when a combinatorial geometry of lines is realized by an honest line arrangement in the projective plane. By a famous result of Mnëv, this latter realization problem is essentially unconstrained: the realization spaces of rank 3 matroids can be arbitrary algebraic sets.

We establish a piecewise linear analog of Mnëv's universality principle, drawing as inspiration von Staudt's "algebra of throws". We define a class of Newton subdivisions of the simplex, **linear Newton subdivisions**, which are candidates for subdivisions arising from line arrangements. We prove a universality theorem concerning such subdivisions.

Theorem C *(Tropical universality). For any polyhedral subset S of $\mathbb{R}^m_{>0}$ defined by linear equalities and inequalities with integer coefficients, there exists a linear Newton subdivision whose realization space is linearly isomorphic to S.*

1.1. Combinatorial incidence geometry

The Sylvester–Gallai theorem states that any collection of non-collinear points in $\mathbb{R}^2$ determines a line passing through exactly two of the points. This theorem has its origins in a problem posed by James Joseph Sylvester in 1893. Tibor Gallai gave an elegant proof of the statement in 1943. It is believed that Sylvester's interest in the question derived from the classical geometry of plane curves. If $C \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane cubic, an explicit calculation shows that C has precisely 9 inflection points. Each line passing through two of these points contains a third. In other words, the Sylvester–Gallai theorem is false over