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Green-to-red sequences for positroids

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A B S T R A C T

Ⅎ-diagrams are combinatorial objects that parametrize cells of the totally nonnegative Grassmannian, called positroid cells, and each Ⅎ-diagram gives rise to a cluster algebra which is believed to be isomorphic to the coordinate ring of the corresponding positroid variety. We study quivers arising from these diagrams and show that they can be constructed from the well-behaved quivers associated to Grassmannians by deleting and merging certain vertices. Then, we prove that quivers coming from arbitrary Ⅎ-diagrams, and more generally reduced plabic graphs, admit a particular sequence of mutations called a green-to-red sequence.

1. Introduction

Since their introduction in 2001, a lot of attention has been paid to Fomin and Zelevinsky's cluster algebras, a class of commutative rings with a distinguished set of generators which are related to each other by a collection of relatively simple combinatorial rules. Many previously well-studied rings, including Grassmannians and double Bruhat cells in flag varieties [21,2], turn out to be cluster algebras. Cluster algebras come with a

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great deal of extra structure, such as a canonical basis and a well-behaved notion of positivity.

A part of the combinatorial framework of many cluster algebras is encoded in an operation called *quiver mutation*, which is a simple transformation performed on a directed graph. In the cases where it is relevant, numerous properties of the cluster algebra can be read off from the quiver alone. For example, skew-symmetric cluster algebras of finite type and finite mutation type are classified via their quivers, see [7,9].

Another important characteristic of a quiver is the existence of a particular sequence of mutations, called a *green-to-red sequence*. It was introduced in [12] under the name *reddening sequence* as a combinatorial tool for computing refined Donaldson–Thomas invariants. The existence of such a sequence has important consequences [18, Sec. 4.4] for the structure of the *scattering diagram*, a recent construction appearing in [11] which was instrumental in resolving several long-standing conjectures in the field. In particular, it follows that the Enough Global Monomials property of [11] holds whenever these sequences exist. Furthermore, in all known cases quivers admitting a *maximal green sequence*, a particular type of a green-to-red sequence, also have the property that their cluster algebra agrees with their upper cluster algebra. See [5] for a detailed overview, however a precise relationship between maximal green sequences and upper cluster algebra remains unclear. Later, it was shown in [18] that the existence of such a sequence is not invariant under mutation, even though the resulting algebras are isomorphic. On the other hand, the existence of a green-to-red sequence is preserved under mutation [18] which offers a more precise characterization of the corresponding cluster algebras.

Next, we briefly recall some of the relevant geometric objects. Given two nonnegative integers $d \leq n$ and a field k, write $Gr_{d,n}(k)$ for the Grassmannian of d-planes in k^n. Given a d-plane $V \subseteq k^n$, we may represent it as an $n \times d$ matrix M by choosing any linear map $k^d \to k^n$ whose image is V. Recall that, given any d-element subset S of the set $\{1, \dots, n\}$, we may define the corresponding *Plücker coordinate* by taking the determinant of the submatrix of M containing only the rows in S. Choosing a different M to represent the same d-plane V amounts to multiplying M on the right by an element of GL_d, which in turn multiplies all the Plücker coordinates by the same nonzero scalar. The Plücker coordinates thereby comprise a set of projective coordinates on the Grassmannian itself.

Given any point in the Grassmannian, we say the *matroid* of that point is the set of Plücker coordinates that are nonzero, and the corresponding *open matroid variety* is the set of all points of the Grassmannian with the same matroid as the chosen point. The closure of the open matroid variety is just called the *matroid variety*.

When $k = \mathbb{R}$, we can speak of the *totally non-negative Grassmannian*, which is the set of all points of $Gr_{d,n}(\mathbb{R})$ for which all Plücker coordinates are nonnegative. The matroid of a point in the totally non-negative Grassmannian is called a *positroid*. For any matroid there is a unique smallest positroid containing it, which we call its *positroid envelope*. We can use this to define the *open positroid variety* corresponding to any positroid P: