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Enumeration of partitions with prescribed successive rank parity blocks

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Successive ranks of a partition, which were introduced by Atkin, are the difference of the lengths of the i-th row and the i-th column in the Ferrers graph. Recently, in the study of singular overpartitions, Andrews revisited successive ranks and parity blocks. Motivated by his work, we investigate partitions with prescribed successive rank parity blocks. The main result of this paper is the generating function of partitions with exactly d successive ranks and m parity blocks.

1. Introduction

In 1944, defining the rank of a partition as the largest part minus the number of parts, F. Dyson conjectured that the rank statistic would account for the following Ramanujan partition congruences combinatorially [10]:

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$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$

where $p(n)$ denotes the number of partitions of n. Dyson's conjecture was proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [8]. In search of the crank statistic for

$$p(11n+6) \equiv 0 \pmod{11},$$

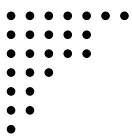
whose existence was conjectured by Dyson, Atkin introduced successive ranks and showed how they would replace the rank statistic in various algebraic expressions [7].

Developing a new sieve method for counting partitions, G.E. Andrews investigated generating functions related to successive ranks [1]. Subsequently Andrews' work was generalized by D.M. Bressoud [9].

In [1] and [9], the concept of oscillations of the successive ranks were introduced and used to study Rogers–Ramanujan type partition identities, and further study was made on variations of successive ranks in [5]. Recently, in the study of singular overpartitions [4], Andrews revisited successive ranks and parity blocks.

The main purpose of this article is to study partitions with prescribed successive ranks and their parity blocks. It turns out that the enumeration of such partitions involves counting a certain type of plane partitions.

A partition may be represented by a Ferrers graph [2]. For instance, the partition $7+5+5+3+2+2+1$ has the representation:



The successive ranks of a partition are defined along the main diagonal of the Ferrers graph, namely the i-th rank is the number of dots in row i minus the number of dots in column i.

Meanwhile, in the Ferrers graph, by reading off rows to the right of (resp. columns below) the main diagonal and putting their sizes on the top row (resp. bottom row), we can represent a partition λ of n as a two-rowed array [3,15]:

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_d \\ y_1 & y_2 & \cdots & y_d \end{pmatrix},$$

where d is the number of dots in the main diagonal, $\sum_{i=1}^{d}(x_i + y_i + 1) = n$, $x_1 > x_2 > \cdots > x_d \geq 0$, and $y_1 > y_2 > \cdots > y_d \geq 0$. This is called the Frobenius symbol of λ. The Frobenius symbol of the partition $7+5+5+3+2+2+1$ is