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Subalgebras of Solomon's descent algebra based on alternating runs

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The number of alternating runs is a natural permutation statistic. We show it can be used to define some commutative subalgebras of the symmetric group algebra, and more precisely of the descent algebra. The Eulerian peak algebras naturally appear as subalgebras of our run algebras. We also calculate the orthogonal idempotents for run algebras in terms of noncommutative symmetric functions.

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1. Introduction

An *alternating run* of a permutation $\sigma \in \mathfrak{S}_n$ is a maximal monotone sequence of consecutive elements in $\sigma_1, \ldots, \sigma_n$. For example, the alternating runs of $\sigma = 14523687$ are 145, 52, 2368, and 87. We denote by $\operatorname{run}(\sigma)$ the number of alternating runs of σ. The enumeration of permutations refined by the number of alternating runs have been the subject of various previous works, beginning with André [3] as early as the late 19th century, then Carlitz in the 70's [9–12]. See also [6,8,13,15,20,21,31] for more recent references. It is worth mentioning that the work [15] was motivated by computations in quantum field theory. Also, alternating runs play a role in some algorithms such as pattern matching [7].

However, we take here a different perspective, our goal being to define and study various algebras based on this notion of alternating runs. Such an algebra first arose in Doyle and Rockmore's study of "ruffle" card-shuffling: [14, Section 5.5] shows that the elements

$$W_k = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{run}(\sigma)=k}} \sigma,$$

for $1 \leq k \leq n-1$, linearly span a commutative subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$, called the *reduced turning algebra*. Doyle and Rockmore also considered the number of alternating runs when we add an initial 0 in front of the permutation and obtain in this way another algebra, called the *turning algebra* [14, Section 5.4]. We will in particular obtain new proofs of these results in the present article.

In fact, it is easily seen that $\operatorname{run}(\sigma)$ only depends on the descent set of σ, so that the elements W_k lie in the descent algebra $\mathcal{D}_n \subseteq \mathbb{Z}[\mathfrak{S}_n]$ (see [27] or the next section for a definition). This widely studied algebra provides other examples of combinatorial statistics such that there is an algebra linearly spanned by sums of permutations having the same value. Consider for example the descent statistic, then the n elements

$$E_k = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}(\sigma)=k}} \sigma,$$

for $0 \leq k \leq n-1$ linearly span a subalgebra of $\mathcal{D}_n$ called the *Eulerian algebra* $\mathcal{E}_n$ (see [17, 19]). But those being particularly relevant here are the Eulerian peak algebra and its left peak analogue [2,23–25], because peaks and alternating runs are tightly connected.

Our main goal is to introduce a new algebra based on alternating runs, as the linear span of the $2n-2$ elements:

$$W_k^+ := \sum_{\substack{\sigma \in \mathfrak{S}_n \;\; \operatorname{run}(\sigma)=k, \\ \text{the first run is ascending}}} \sigma, \qquad W_k^- := \sum_{\substack{\sigma \in \mathfrak{S}_n \;\; \operatorname{run}(\sigma)=k, \\ \text{the first run is descending}}} \sigma,$$