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# Poset edge densities, nearly reduced words, and barely set-valued tableaux

Victor Reiner [a], Bridget Eileen Tenner [b], Alexander Yong [c]

[a] *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA*
[b] *Department of Mathematical Sciences, DePaul University, Chicago, IL 60614, USA*
[c] *Department of Mathematics, University Illinois at Urbana-Champaign, Urbana, IL 61801, USA*

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ABSTRACT

In certain finite posets, the expected down-degree of their elements is the same whether computed with respect to either the uniform distribution or the distribution weighting an element by the number of maximal chains passing through it. We show that this coincidence of expectations holds for Cartesian products of chains, connected minuscule posets, weak Bruhat orders on finite Coxeter groups, certain lower intervals in Young's lattice, and certain lower intervals in the weak Bruhat order below dominant permutations. Our tools involve formulas for counting nearly reduced factorizations in 0-Hecke algebras; that is, factorizations that are one letter longer than the Coxeter group length.
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## 1. Introduction

The *edge density* of a finite poset $P$ is the ratio of the number of its covering relations $q \lessdot p$ to its cardinality $\#P$. One can also interpret this ratio as the expectation $\mathbb{E}(X)$ of

*E-mail addresses:* reiner@math.umn.edu (V. Reiner), bridget@math.depaul.edu (B.E. Tenner), ayong@illinois.edu (A. Yong).

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a random variable $X$ on $P$, counting the elements covered by $p \in P$. That is, the random variable $X$ computes the down-degree of $p$ in the Hasse diagram of $P$, with respect to the uniform distribution.

If, instead, one changes this distribution by assigning to each $p \in P$ a probability proportional to the number of maximal chains through $p$ in $P$, then one can define a random variable $Y$ on $P$ whose value is again the down-degree of $p$ in the Hasse diagram, but now weighted by that probability.

Given the different distributions in play, one would generally not expect the expectations for $X$ and $Y$ to be equal. However, we prove that, in a variety of interesting settings, one does indeed find equality, and we conjecture that equality holds in several additional settings, as well.

**Definition.** A finite poset $P$ has *coincidental down-degree expectations (CDE)* if $\mathbb{E}(X) = \mathbb{E}(Y)$.

We may also refer to $P$ as *being* CDE. This terminology will be made more precise in Definition 2.1. To motivate our study, consider the following examples of CDE posets.

- *Disjoint unions of chains* are CDE because the two probability distributions are the same in this setting.
- *Cartesian products of chains* are CDE because Proposition 2.13 will show that CDE is preserved under Cartesian products of graded posets.
- *Weak Bruhat order* on a finite Coxeter group is CDE. In fact, any weak order on the chambers of a (central, essential) *simplicial hyperplane arrangement* in $\mathbb{R}^r$ (or, more generally, the topes of an *oriented matroid* of rank $r$) is CDE, as will be shown in Corollary 2.21.
- *Tamari lattices* on polygon triangulations are CDE, as will be shown in Corollary 2.22.
- *Connected minuscule posets* are CDE, as will be shown in Theorem 2.10. Also the distributive lattices $J(P)$ associated to arbitrary minuscule posets $P$ are CDE, as will be shown in Theorem 2.11.
- Our main result, Theorem 1.1, exhibits a rich class of lower intervals in *Young's lattice* and in weak Bruhat orders on permutations, all of which are CDE. (In fact, this paper grew from an attempt to understand Corollary 1.3 of Theorem 1.1 in two different ways.)

Before stating our main result, we recall a few definitions. *Young's lattice* is the partial order on integer partitions $\lambda$ according to containment of their *Ferrers diagrams* $\mu \subset \lambda$. The *(right) weak Bruhat order* on permutations in the symmetric group $\mathfrak{S}_n$ is the transitive closure of the relation $u \lessdot w$ if $w = us$ for some adjacent transposition $s = \sigma_i = (i, i+1)$ with $u(i) < u(i+1)$. A permutation $w = w(1) \cdots w(n) \in \mathfrak{S}_n$ is *vexillary* if it is 2143-avoiding; that is, if there are no quadruples $i_1 < i_2 < i_3 < i_4$ with

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