Journal of Combinatorial Theory, Series A 158 (2018) 126–175

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Oriented flip graphs of polygonal subdivisions and noncrossing tree partitions

Alexander Garver, Thomas McConville

ARTICLE INFO

Article history:
Received 2 February 2017

Keywords:
Simplicial complex
Lattice
Noncrossing partition
Shard intersection order

ABSTRACT

Given a tree embedded in a disk, we introduce a simplicial complex of noncrossing geodesics supported by the tree, which we call the noncrossing complex. The facets of the noncrossing complex have the structure of an oriented flip graph. Special cases of these oriented flip graphs include the Tamari lattice, type A Cambrian lattices, Stokes posets of quadrangulations, and oriented exchange graphs of quivers mutation-equivalent to a type A Dynkin quiver. We prove that the oriented flip graph is a polygonal, congruence-uniform lattice. To do so, we express the oriented flip graph as a lattice quotient of a lattice of biclosed sets.

The facets of the noncrossing complex have an alternate ordering known as the shard intersection order. We prove that this shard intersection order is isomorphic to a lattice of noncrossing tree partitions, which generalizes the classical lattice of noncrossing set partitions. The oriented flip graph inherits a cyclic action from its congruence-uniform lattice structure. On noncrossing tree partitions, this cyclic action generalizes the classical Kreweras complementation on noncrossing set partitions.

Contents

E-mail address: alexander.garver@gmail.com (A. Garver).

https://doi.org/10.1016/j.jcta.2018.03.014
0097-3165/

1. Introduction

The purpose of this work is to understand the combinatorics associated with lattices of polygonal subdivisions (equivalently, partial triangulations) of a convex polygon. We refer to the lattices of polygonal subdivisions we study as **oriented flip graphs** (see Definition 3.10). Special cases of these posets include the Tamari lattice, type A Cambrian lattices [28], oriented exchange graphs of type A cluster algebras [5], and the Stokes poset of quadrangulations defined by Chapoton [8]. As a consequence of our work, we prove and generalize Chapoton's conjecture claiming that the Stokes poset is a lattice [8].

Rather than directly studying polygonal subdivisions, it turns out to be more convenient to formulate our theory in terms of trees that are dual to polygonal subdivisions of a polygon. That is, our work begins with the initial data of a tree T embedded in a disk so that its leaves lie on the boundary and its other vertices lie in the interior of the disk. This data gives rise to a simplicial complex of **noncrossing** sets of **arcs** on this tree that we call the **noncrossing complex**, $\Delta^{NC}(T)$. One of our main results is that the noncrossing complex is a pure and thin simplicial complex (see Corollaries 3.6 and 3.9). The combinatorics of $\Delta^{NC}(T)$ allow us to define our oriented flip graphs, which we denote by $\overrightarrow{FG}(T)$.

After defining oriented flip graphs, we turn our attention to understanding their lattice theoretic aspects. In Theorem 4.14, we show that for any tree T, the oriented flip graph $\overrightarrow{FG}(T)$ is a **congruence-uniform lattice**. In particular, any oriented flip graph is a lattice.

The Tamari lattice is a standard example of a congruence-uniform lattice [20]; see also [7], [28]. Reading gave a proof of congruence-uniformity of the Tamari lattice by proving that the weak order on permutations is congruence-uniform and applying the lattice quotient map from the weak order to the Tamari lattice defined by Björner and