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Bijections for planar maps with boundaries

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ABSTRACT

We present bijections for planar maps with boundaries. In particular, we obtain bijections for triangulations and quadrangulations of the sphere with boundaries of prescribed lengths. For triangulations we recover the beautiful factorized formula obtained by Krikun using a (technically involved) generating function approach. The analogous formula for quadrangulations is new. We also obtain a far-reaching generalization for other face-degrees. In fact, all the known enumerative formulas for maps with boundaries are proved bijectively in the present article (and several new formulas are obtained).

Our method is to show that maps with boundaries can be endowed with certain "canonical" orientations, making them amenable to the master bijection approach we developed in previous articles. As an application of our enumerative formulas, we note that they provide an exact solution of the dimer model on rooted triangulations and quadrangulations.

1. Introduction

In this article, we present bijections for planar maps with boundaries. Recall that a *planar map* is a decomposition of the 2-dimensional sphere into vertices, edges, and faces,

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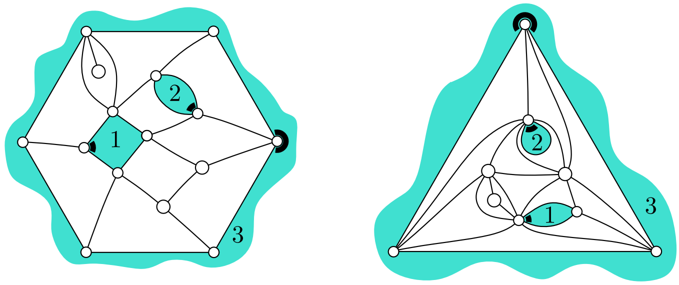

Fig. 1. Left: a quadrangulation in $\mathcal{Q}[3;4,2,6]$. Right: a triangulation in $\mathcal{T}[3;2,1,3]$.

considered up to continuous deformation (see precise definitions in Section 2). We deal exclusively with *planar* maps in this article and call them simply *maps* from now on. A *map with boundaries* is a map with a set of distinguished faces called *boundary faces* which are pairwise vertex-disjoint, and have simple-cycle contours (no pinch points). We call *boundaries* the contours of the boundary faces. We can think of the boundary faces as holes in the sphere, and maps with boundaries as a decomposition of a sphere with holes into vertices, edges and faces. A *triangulation with boundaries* (resp. *quadrangulation with boundaries*) is a map with boundaries such that every non-boundary face has degree 3 (resp. 4).

The main results obtained in this article are bijections for triangulations and quadrangulations with boundaries. The bijection establishes a correspondence between these maps and certain types of plane trees. This, in turns, easily yields factorized enumeration formulas with control on the number and lengths of the boundaries. In the case of triangulations, the enumerative formula had been established by Krikun [10] (by a technically involved "guessing/checking" generating function approach). The case of quadrangulations is new. We also present a far-reaching generalization for maps with other face-degrees.

The strategy we apply is to adapt to maps with boundaries the "master bijection" approach we developed in [2,3] for maps without boundaries. Roughly speaking, this strategy reduces the problem of finding bijections, to the problem of exhibiting canonical orientations characterizing these classes of maps.

Let us now state the enumerative formulas derived from our bijections for triangulations and quadrangulations. We call a map with boundaries *multi-rooted* if the r boundary faces are labeled with distinct numbers in $[r] = \{1,\dots,r\}$, and each one has a marked corner; see Fig. 1. For $m \geq 0$ and $a_1,\dots,a_r$ positive integers, we denote $\mathcal{T}(m;a_1,\dots,a_r)$ (resp. $\mathcal{Q}(m;a_1,\dots,a_r)$) the set of multi-rooted triangulations (resp. quadrangulations) with r boundary faces, and m internal vertices (vertices not on the boundaries), such that the boundary labeled i has length a_i for all $i \in [r]$. In 2007 Krikun proved the following result: