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Kronecker coefficients and noncommutative super Schur functions

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ABSTRACT

The theory of noncommutative Schur functions can be used to obtain positive combinatorial formulae for the Schur expansion of various classes of symmetric functions, as shown by Fomin and Greene [11]. We develop a theory of noncommutative super Schur functions and use it to prove a positive combinatorial rule for the Kronecker coefficients $g_{\lambda\mu\nu}$ where one of the partitions is a hook, recovering previous results of the two authors [7,22]. This method also gives a precise connection between this rule and a heuristic for Kronecker coefficients first investigated by Lascoux [19].

1. Introduction

Let M_λ be the irreducible representation of the symmetric group $\mathcal{S}_n$ corresponding to the partition λ. Given three partitions λ, μ, and ν of n, the *Kronecker coefficient* $g_{\lambda\mu\nu}$ is the multiplicity of M_ν in the tensor product $M_\lambda \otimes M_\mu$. A longstanding open problem in

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algebraic combinatorics, called the *Kronecker problem*, is to find a positive combinatorial formula for these coefficients. See [2,7,9,10,16,19,22–26] for some known special cases.

Our story begins with the work of Lascoux [19], wherein he gave a formula for the Kronecker coefficients $g_{\lambda\mu\nu}$ when two of the partitions are hooks by considering products of permutations in certain Knuth equivalence classes. Though this rule no longer holds outside the hook-hook case, it seems to approximate Kronecker coefficients amazingly well for *any three partitions* and therefore gives a useful heuristic.

Several years ago, the first author [7] gave a rule for Kronecker coefficients when one of the partitions is a hook. This rule was discovered using Lascoux's heuristic, but it was left as an open problem to give a precise statement relating it to the heuristic. Recently, the second author [22] gave a simplified description and proof of this rule.

We develop a theory of noncommutative super Schur functions based on work of Fomin–Greene [11] and the first author [5]. Using this we

- reprove and strengthen the rule from [22],
- establish a precise connection between this rule and the Lascoux heuristic, and
- uncover a surprising parallel between this rule and combinatorics underlying transformed Macdonald polynomials indexed by a 3-column shape, as described in [14,5].

2. Main results

In this section, we state our main theorem on noncommutative super Schur functions (Theorem 2.3), and show how it can be used to recover the rule from [22] for Kronecker coefficients where one of the partitions is a hook. Proofs will be deferred to later sections.

2.1. Colored words and the algebra $\mathcal{U}$

Let $\mathcal{A}_\varnothing = \{1, 2, \ldots, N\}$ denote the alphabet of unbarred letters and $\mathcal{A}_- = \{\bar{1}, \bar{2}, \ldots, \overline{N}\}$ the alphabet of barred letters. A *colored word* is a word in the total alphabet $\mathcal{A} = \mathcal{A}_\varnothing \sqcup \mathcal{A}_-$.

We will consider total orders $\lessdot$ on $\mathcal{A}$ such that $1 \lessdot 2 \lessdot \cdots \lessdot N$ and $\bar{1} \lessdot \bar{2} \lessdot \cdots \lessdot \overline{N}$; we call such orders *shuffle orders*. Two shuffle orders we will work with frequently are

$$\begin{aligned} &\text{the } \textit{natural order} < \text{ given by} \quad 1 < \bar{1} < 2 < \bar{2} < \cdots < N < \overline{N}, \qquad \text{and} \\ &\text{the } \textit{big bar order} \prec \text{ given by} \quad 1 \prec 2 \prec \cdots \prec N \prec \bar{1} \prec \bar{2} \prec \cdots \prec \overline{N}. \end{aligned}$$

Let $\mathcal{U}$ be the free associative $\mathbb{Z}$-algebra in the noncommuting variables u_x, $x \in \mathcal{A}$. Equivalently, $\mathcal{U}$ is the tensor algebra of $\mathbb{Z}\mathcal{A}$, the $\mathbb{Z}$-module freely spanned by the elements of $\mathcal{A}$. We identify the monomials of $\mathcal{U}$ with colored words and frequently write x for the variable u_x and $\mathsf{w} = \mathsf{w}_1 \cdots \mathsf{w}_t = u_{w_1} \cdots u_{w_t}$ for a colored word/monomial.

For the natural order $<$, it is useful to have a notation for "going down by one." Accordingly, for any $x \in \mathcal{A}$, define