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A weighted cellular matrix-tree theorem, with applications to complete colorful and cubical complexes ☆,☆☆

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A R T I C L E I N F O

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A B S T R A C T

We present a version of the weighted cellular matrix-tree theorem that is suitable for calculating explicit generating functions for spanning trees of highly structured families of simplicial and cell complexes. We apply the result to give weighted generalizations of the tree enumeration formulas of Adin for complete colorful complexes, and of Duval, Klivans and Martin for skeleta of hypercubes. We investigate the latter further via a logarithmic generating function for weighted tree

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enumeration, and derive another tree-counting formula using the unsigned Euler characteristics of skeleta of a hypercube.

1. Introduction

The matrix-tree theorem, first discovered by Kirchhoff in 1845, expresses the number of spanning trees of a (finite, undirected) graph in terms of the spectrum of its Laplacian matrix. It can be used to derive closed formulas for the spanning tree counts of numerous families of graphs such as complete, complete bipartite, complete multipartite, hypercube and threshold graphs; see, e.g., [22, §5.6] and [19, Chapter 5]. The matrix-tree theorem has a natural weighted analogue that expresses the generating function for spanning trees in terms of the spectrum of a weighted Laplacian matrix. For certain graphs with tight internal structure, the associated tree generating functions for statistics such as degree sequence have explicit factorizations which can be found by examination of the weighted Laplacian spectra.

Central to the matrix-tree theorem is the characterization of a spanning tree of a graph as a set of edges corresponding to a column basis of its incidence matrix. This observation holds true in the more general context of finite simplicial and CW complexes, an idea introduced by Bolker [3] and Kalai [14] and recently studied by many authors; see [10] for a survey. The matrix-tree theorem and its weighted versions extend to this broader context, raising the question of finding explicit formulas for generating functions for spanning trees in highly structured CW complexes. Specifically, for a CW complex Δ of dimension $\geq k$, let $\{X_\sigma\}$ be a set of commuting indeterminates corresponding to the cells $\sigma \in \Delta$, let $\mathcal{T}_k(\Delta)$ denote its set of k-dimensional spanning trees, and let $\tilde{H}$ denote reduced cellular homology. The higher-dimensional analogue of the (unweighted) tree count is

$$\tau_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}_k(\Delta)} |\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})|^2$$

and the corresponding generating function (the *weighted tree count*) is

$$\hat{\tau}_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}_k(\Delta)} |\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})|^2 \prod_{\sigma \in \Upsilon_k} X_\sigma;$$

the homology-squared summands in each case arise from the proof of the matrix-tree theorem [14,8], and each summand simply equals 1 when $k = 1$.

The indeterminates X_σ may be further specialized. Kalai [14] calculated τ_k and $\hat{\tau}_k$ for *skeleta of simplices* (see (2) and (4) below), respectively generalizing Cayley's formula n^{n-2} for the graph case ($k = 1$) and the degree-sequence generating function that can