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The free-fermionic $C_2^{(1)}$ loop model, double dimers and Kashaev's recurrence

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A B S T R A C T

We study a two-color loop model known as the $C_2^{(1)}$ loop model. We define a free-fermionic regime for this model, and show that under this assumption it can be transformed into a double dimer model. We then compute its free energy on periodic planar graphs. We also study the star-triangle relation or Yang–Baxter equations of this model, and show that after a proper parametrization they can be summed up into a single relation known as Kashaev's relation. This is enough to identify the solution of Kashaev's relation as the partition function of a $C_2^{(1)}$ loop model with some boundary conditions, thus solving an open question of Kenyon and Pemantle [29] about the combinatorics of Kashaev's relation.

1. Introduction

In 1996 Kashaev introduced a way to rewrite the star triangle transformation of the Ising model [24]. Specifically, let us take a planar graph $G = (V, E)$ with usual coupling constants for the Ising model $(J_e)_{e\in E}$ on the edges. Let us suppose that there is a set of variables g on the vertices and faces of G such that

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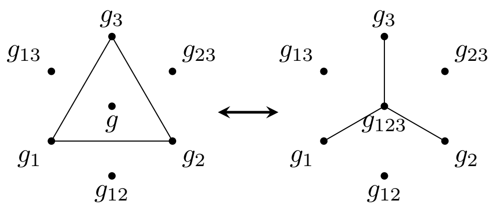
$$\sinh^2(J_e) = \frac{g_x g_y}{g_u g_v} \tag{1}$$

where x, y are the endpoints of e and u, v are the faces adjacent to e. Then the star-triangle relation, or local Yang–Baxter equation, is equivalent to the variables g satisfying a single polynomial relation:



$$\begin{aligned}
&g^2 g_{123}^2 + g_1^2 g_{23}^2 + g_2^2 g_{13}^2 + g_3^2 g_{12}^2 \\
&-2g_2 g_3 g_{13} g_{12} - 2g_1 g_3 g_{23} g_{12} - 2g_1 g_2 g_{23} g_{13} \\
&-2g g_{123}(g_1 g_{23} + g_2 g_{13} + g_3 g_{12}) \\
&-4g g_{12} g_{23} g_{13} - 4g_{123} g_1 g_2 g_3 \\
&=0.
\end{aligned}$$

This relation,[1] known as Kashaev's relation, has sparked some interest from the point of view of *spatial recurrences*. It can be embedded in $\mathbb{Z}^3$ by taking $x \in \mathbb{Z}^3$ and denoting $g = g_x$, $g_i = g_{x+e_i}$, $g_{ij} = g_{x+e_i+e_j}$, etc. Then by choosing the greatest root of a degree 2 polynomial we get [29]:

$$g_{123} = \frac{2g_1 g_2 g_3 + g(g_1 g_{23} + g_2 g_{13} + g_3 g_{12}) + 2XYZ}{g^2}, \tag{2}$$

where $X = \sqrt{g g_{23} + g_2 g_3}$, $Y = \sqrt{g g_{13} + g_1 g_3}$, $Z = \sqrt{g g_{12} + g_1 g_2}$.

This transformation (2) is called *Kashaev's recurrence*. It can be iterated to define g on further vertices of $\mathbb{Z}^3$, provided we had a sufficiently large set of initial conditions. A remarkable fact is that it exhibits a *Laurentness* phenomenon: the solution of the recurrence at any point is always a Laurent polynomial in the initial variables. This fact is related to cluster algebras [17,18], but it also hints at a possible hidden object represented by the solution.

Let us quickly review the current state of spatial recurrences: Speyer related the solution of the octahedron recurrence (which can be traced back to Dodgson [12]) to the partition function of a dimer model [35]; then Carroll and Speyer showed that the cube recurrence (proposed by Propp [34]) corresponds to cube groves [5]; more recently Kenyon and Pemantle studied a generalization of Kashaev's relation, known as the hexahedron recurrence, and identified its solution with a double dimer model [29]. Unfortunately, when specialized to Kashaev's recurrence, their model does not provide a one-to-one correspondence between configurations and monomials of the Laurent polynomial. In this paper we provide a model that does give a one-to-one correspondence, known in the physics literature as the $C_2^{(1)}$ loop model.

[1] Kashaev's initial equation contained a $+4$ instead of a -4 coefficient for the last terms, but one can easily get from one to another, for instance by multiplying g by -1 at a vertex of the cube and its three neighbors.