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Extremal hypergraphs for Ryser's Conjecture

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ABSTRACT

Ryser's Conjecture states that any r-partite r-uniform hypergraph has a vertex cover of size at most $r-1$ times the size of the largest matching. For $r=2$, the conjecture is simply König's Theorem and every bipartite graph is a witness for its tightness. The conjecture has also been proven for $r=3$ by Aharoni using topological methods, but the proof does not give information on the extremal 3-uniform hypergraphs. Our goal in this paper is to characterize those hypergraphs which are tight for Aharoni's Theorem.

Our proof of this characterization is also based on topological machinery, particularly utilizing results on the (topological) connectedness of the independence complex of the line graph of the link graphs of 3-uniform Ryser-extremal hypergraphs. We use this information to nail down the elements of a structure we call *home-base hypergraph*. While there is a single minimal home-base hypergraph with matching number k for every positive integer $k \in \mathbb{N}$, home-base hypergraphs with matching number k are far from being unique. There are infinitely many of them and each of them is composed of k copies of two different kinds of basic structures, whose hyperedges can intersect in various restricted, but intricate ways.

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Our characterization also proves an old and wide open strengthening of Ryser's Conjecture, due to Lovász, for the 3-uniform extremal case, that is, for hypergraphs with $\tau = 2\nu$.

1. Introduction

A hypergraph $\mathcal{H}$ is a pair (V, E), where $V = V(\mathcal{H})$ is the set of *vertices*, and $E = E(\mathcal{H})$ is a **multiset** of subsets of vertices called the *edges* of $\mathcal{H}$. The number of times a subset $e \subseteq V$ appears in E is called the *multiplicity* of e. If the cardinality of every edge is r, we call $\mathcal{H}$ an *r-graph*. A 2-graph is called a *graph*. In our paper we mostly have no restriction on the multiplicity of edges; whenever we want to assume that each multiplicity is at most 1, we will explicitly say *simple hypergraph*, *simple r-graph*, or *simple graph*. An edge $e \in E$ is called *parallel* to an edge $f \in E$ if their underlying vertex subsets are the same. In particular, every edge is parallel to itself.

Let $\mathcal{H}$ be a hypergraph. A *matching* in $\mathcal{H}$ is a set of disjoint edges of $\mathcal{H}$, and the *matching number*, $\nu(\mathcal{H})$, is the size of the largest matching in $\mathcal{H}$. If $\nu(\mathcal{H}) = 1$, then $\mathcal{H}$ is called *intersecting*. A *vertex cover* of $\mathcal{H}$ is a set of vertices which intersects every edge of $\mathcal{H}$. The size of the smallest vertex cover is called the *vertex cover number* of $\mathcal{H}$ and is denoted by $\tau(\mathcal{H})$. It is immediate to see that if $\mathcal{H}$ is r-uniform, then the following bounds always hold:

$$\nu(\mathcal{H}) \leq \tau(\mathcal{H}) \leq r\nu(\mathcal{H}).$$

Both inequalities are easily seen to be tight for general hypergraphs. Ryser's Conjecture [16], which appeared first in the late 1960's, states that the upper bound can be lowered by considering only r-partite hypergraphs. An r-graph is called *r-partite* if its vertices can be partitioned into r parts, called *vertex classes*, such that every edge intersects each vertex class in exactly one vertex.

Conjecture 1 *(Ryser's Conjecture). If $\mathcal{H}$ is an r-partite r-graph, then*

$$\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H}).$$

Around the same time a much stronger conjecture was made by Lovász [13]. The conjecture states that not only do we have a vertex cover of size $(r-1)\nu(\mathcal{H})$, but we can obtain it by repeatedly reducing the matching number by one with the removal of $r-1$ vertices.

Conjecture 2 *(Lovász conjecture). In every r-partite r-graph there exist $r-1$ vertices whose deletion reduces the matching number.*