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Monochromatic solutions to systems of exponential equations

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ABSTRACT

Let $n \in \mathbb{N}$, R be a binary relation on $[n]$, and $C_1(i,j),\dots,C_n(i,j) \in \mathbb{Z}$, for $i,j \in [n]$. We define the exponential system of equations $\mathcal{E}(R,(C_k(i,j))_{i,j,k})$ to be the system

$$X_i^{Y_1^{C_1(i,j)} \cdots Y_n^{C_n(i,j)}} = X_j, \text{ for } (i,j) \in R,$$

in variables $X_1,\dots,X_n,Y_1,\dots,Y_n$. The aim of this paper is to classify precisely which of these systems admit a monochromatic solution $(X_i, Y_i \neq 1)$ in an arbitrary finite colouring of the natural numbers. This result could be viewed as an analogue of Rado's theorem for exponential patterns.

1. Introduction

In 2011, Sisto [16] made the surprising observation that an arbitrary 2-colouring of the natural numbers admits infinitely many integers $a, b > 1$ such that a, b, a^b all receive the same colour. He went on to ask if a similar result holds for k-colourings of the natural numbers with $k > 2$. Brown [3], simplifying and extending the proof of Sisto, gave further examples of exponential, monochromatic patterns that are present in an arbitrary 2-colouring and also proved some weaker results for monochromatic patterns in more

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colours. In a recent paper [15] we answered Sisto's question by showing that any *finite* colouring of the positive integers admits $a, b > 1$ such that $\{a, b, a^b\}$ is monochromatic and went on to develop, in this context, a theory of patterns defined by compositions of the exponential function. In the present paper we turn from the study of patterns arising as compositions of the exponential function, to understand exponential patterns that arise as solutions to systems of equations. There is a vast literature on finding patterns in arbitrary finite partitions of the integers, [1,2,4–9,12,17,13,19] and we refer the reader to the introduction in [15] for a brief discussion of this theory or to [6] for a indepth survey of the most classical elements of the theory.

The motivation for the study of monochromatic solutions to equations lies in the seminal work of Rado [14], who classified the systems of homogeneous linear equations that admit a solution in an arbitrary finite colouring of the natural numbers. More precisely, we say that an $m \times n$ matrix A is *partition regular* if every finite colouring of $\mathbb{N}$ admits monochromatic $x_1, \dots, x_n \in \mathbb{N}$, for which $Ax = 0$, where $x = (x_1, \dots, x_n)$. Rado classified the partition regular matrices by giving a simple criterion on the columns of such matrices. It is in this spirit that the present paper sets out.

It is worth pointing out that, even in the classical, linear theory, there is a distinction between studying patters which solve linear systems and patterns which arise as fixed linear compositions of several free variables. These two types of partition regularity are sometimes termed "kernel partition regular" and "image partition regular", respectively. So, while Rado's theorem gave a complete understanding of what linear systems $Ax = 0$ can be solved in an arbitrary colouring, it was not until the work Hindman and Leader [11] that a classification of "image" partition regular systems was fully understood. We refer the reader to the survey of Hindman [10], for details.

Before going further, let us lay out some basic terminology. Let $k \in \mathbb{N}$ and X be a non-empty set. We call a function $f\colon \mathbb{N} \to X$ a *finite colouring* if X is finite, and a *k-colouring*, if $|X| \le k$. As is standard, we refer to the elements of X as *colours*. We say that a collection $\mathcal{A}$, of ordered tuples of integers, is *partition regular* if for every finite colouring $f\colon \mathbb{N} \to X$ we can find $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{N}$, such that $f(x_1) = \cdots = f(x_n)$ and $(x_1, \dots, x_n) \in \mathcal{A}$. We say that a linear system of equations is *partition regular* if its solution set in $\mathbb{N}$ is partition regular. We say that an exponential system of equations is *partition regular* if its solution set in $\mathbb{N} \setminus \{1\}$ is partition regular. That is, for exponential equations, we only consider solutions where each coordinate at least 2, to remove the trivial cases. It shall also be convenient to define the binary operation $\star$ as $a \star b = a^b$, for $a, b \in \mathbb{N}$.

For $n \in \mathbb{N}$, let R be a binary relation on $[n]$. Given integers $C_1(i,j), \dots, C_n(i,j) \in \mathbb{Z}$, for $i, j \in [n]$, we define the system of equations $\mathcal{E}\left(R, \{C_k(i,j)\}_{i,j,k}\right)$ by

$$X_i^{Y_1^{C_1(i,j)} \cdots Y_n^{C_n(i,j)}} = X_j, \text{ for } (i,j) \in R, \tag{1}$$

where $X_1, \dots, X_n, Y_1, \dots, Y_n$ are variables.