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Lipschitz polytopes of posets and permutation statistics ☆

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A B S T R A C T

We introduce Lipschitz functions on a finite partially ordered set P and study the associated Lipschitz polytope $\mathscr{L}(P)$. The geometry of $\mathscr{L}(P)$ can be described in terms of descent-compatible permutations and permutation statistics that generalize descents and big ascents. For ranked posets, Lipschitz polytopes are centrally-symmetric and Gorenstein, which implies symmetry and unimodality of the statistics. Finally, we define (P, k)-hypersimplices as generalizations of classical hypersimplices and give combinatorial interpretations of their volumes and h^*-vectors.

1. Introduction

Let $(P, \preceq)$ a finite partially ordered set (or **poset**, for short). A function $f : P \to \mathbb{R}$ is **isotone** or **order preserving** if

$$f(a) \leq f(b) \quad \text{whenever} \quad a \preceq b\,.$$

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The **order cone** $\mathcal{K}(P)$ of P is the collection of nonnegative isotone functions. The order cone is a gateway for a geometric perspective on enumerative problems on posets. The interplay of combinatorics and geometry is, in particular, fueled by analogies to *continues* mathematics. For example, Stanley's order polytope [18] is the set

$$\mathcal{O}(P) \ = \ \left\{ f \in \mathcal{K}(P) \ : \ \|f\|_\infty \leq 1 \right\},$$

where $\|f\|_\infty = \max\{f(a) : a \in P\}$. The theory of P-partitions concerns those $f \in \mathcal{K}(P)$ with $\|f\|_1 = \sum_a f(a) = m$ for some fixed m. See [2, Ch. 6] for enumerative consequences of this geometric perspective. In this paper, we want to further the analogies to continuous functions. For two elements $a, b \in P$, we denote the minimal length of a saturated (or unrefineable) chain from a to b by $d_P(a,b)$ and set $d_P(a,b) := \infty$ if $a \not\preceq b$. Then d_P is a *quasi-metric* on P. An isotone function $f : (P, \preceq) \to \mathbb{R}$ is ***k*-Lipschitz** if

$$f(b) - f(a) \ \leq \ k \cdot d_P(a,b)$$

for all $a \preceq b$. We say that a function f is Lipschitz if f is 1-Lipschitz. Let us write $\check{P}$ for the poset obtained from P by adjoining a minimum $\widehat{0}$. The collection $\widetilde{\mathscr{L}}(\check{P})$ of isotone Lipschitz functions on $\check{P}$ is naturally an unbounded polyhedron and k-Lipschitz functions are precisely the elements in $k \cdot \widetilde{\mathscr{L}}(\check{P})$. The lineality space of $\widetilde{\mathscr{L}}(\check{P})$ is given by all constant functions and we define the **Lipschitz polytope** of P as

$$\mathscr{L}(P) \ := \ \left\{ f \in \mathcal{K}(\check{P}) \ : \ f \text{ Lipschitz}, \ f(\widehat{0}) = 0 \right\}.$$

Concretely, the Lipschitz polytope of $(P, \preceq)$ is given by

$$\mathscr{L}(P) \ = \ \left\{ f \in \mathbb{R}^P : \begin{array}{rcll} 0 \leq & f(a) & \leq 1 & \text{for } a \in \min P \\ 0 \leq & f(b) - f(a) & \leq 1 & \text{for } a \prec\!\!\cdot\, b \end{array} \right\}, \tag{1.1}$$

where $a \prec\!\!\cdot\, b$ denotes the cover relations of P.

A different motivation for the study of $\mathscr{L}(P)$ comes from G-Shi arrangements. The Hasse diagram of $\check{P}$ is the directed graph G on nodes $\check{P}$ with arcs (a,b) whenever $a \prec\!\!\cdot\, b$ is a cover relation. The corresponding ***G*-Shi arrangement** is the arrangement of affine hyperplanes $\{x_b - x_a = 0\}$ and $\{x_b - x_a = 1\}$ for $a \prec\!\!\cdot\, b$. The G-Shi arrangements generalize the classical Shi arrangements [16, Ch. 7] and naturally occur in the geometric combinatorics of parking functions and spanning trees; see [8]. The Lipschitz polytope $\mathscr{L}(P)$ is thus a particular (relatively) bounded region of the G-Shi arrangement associated to the Hasse diagram of P.

We give some basic geometric properties of Lipschitz polytopes in Section 2 and, in particular, show that $\mathscr{L}(P)$ is always a lattice polytope. Hence, the function

$$E(\mathscr{L}(P), k) \ := \ \left| k \cdot \mathscr{L}(P) \cap \mathbb{Z}^P \right|$$