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On Boolean intervals of finite groups

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ABSTRACT

We prove a dual version of Øystein Ore's theorem on distributive intervals in the subgroup lattice of finite groups, having a nonzero dual Euler totient $\hat{\varphi}$. For any Boolean group-complemented interval, we observe that $\hat{\varphi} = \varphi \neq 0$ by the original Ore's theorem. We also discuss some applications in representation theory. We conjecture that $\hat{\varphi}$ is always nonzero for Boolean intervals. In order to investigate it, we prove that for any Boolean group-complemented interval $[H, G]$, the graded coset poset $\hat{P} = \hat{C}(H, G)$ is Cohen–Macaulay and the nontrivial reduced Betti number of the order complex $\Delta(P)$ is $\hat{\varphi}$, so nonzero. We deduce that these results are true beyond the group-complemented case with $|G : H| < 32$. One observes that they are also true when H is a Borel subgroup of G.

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1. Introduction

An extension of Øystein Ore's result [11, Theorem 7] into the framework of planar algebras was investigated by the second author. It led to [12, Conjecture 1.6] which admits two group-theoretical translations dual to each other [12, Theorem 6.11]. One of them recovers the original theorem and the other is the dual version which we investigate here.

Throughout the paper, an *interval of finite groups* $[H, G]$ will always mean an interval in the subgroup lattice of the finite group G, with H as a subgroup.

Section 2 consists of some basics (which are freely used in this introduction) about lattices, order complex, Cohen–Macaulay posets, edge labeling and GAP coding. In Section 3, we first prove a generalization of the following Ore's theorem to any top Boolean interval.

Theorem 1.1. *Let $[H, G]$ be a distributive interval of finite groups. Then there exists $g \in G$ such that $\langle H, g\rangle = G$.*

Then we investigate a dual version.

Definition 1.2. Let $[H, G]$ be an interval of finite groups. Its *Euler totient* $\varphi(H, G)$ is the number of cosets Hg such that $\langle Hg\rangle = G$. *Note that* $\langle Hg\rangle = \langle H, g\rangle$.

Similar to Hall's argument in [9], for any $K \in [H, G]$, $\sum_{L\in[H,K]} \varphi(H, L)$ is precisely $|K : H|$, so by Möbius inversion formula,

$$\varphi(H, G) = \sum_{K\in[H,G]} \mu(K, G)|K : H|.$$

Definition 1.3. Let $[H, G]$ be an interval of finite groups. Its *dual Euler totient* is

$$\hat{\varphi}(H, G) := \sum_{K\in[H,G]} \mu(H, K)|G : K|.$$

Let $[T, G]$ be the top interval of $[H, G]$. By the crosscut theorem [16, Corollary 3.9.4], $\mu(L, G) = 0$ for all $L \in [H, G] \setminus [T, G]$, so