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Dual immaculate quasisymmetric functions expand positively into Young quasisymmetric Schur functions

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We describe a combinatorial formula for the coefficients when the dual immaculate quasisymmetric functions are decomposed into Young quasisymmetric Schur functions. We prove this using an analogue of Schensted insertion. Using this result, we give necessary and sufficient conditions for a dual immaculate quasisymmetric function to be symmetric. Moreover, we show that the product of a Schur function and a dual immaculate quasisymmetric function expands positively in the Young quasisymmetric Schur basis. We also discuss the decomposition of the Young noncommutative Schur functions into the immaculate functions. Finally, we provide a Remmel–Whitney-style rule to generate the coefficients of the decomposition of the dual immaculates into the Young quasisymmetric Schurs algorithmically and an analogous rule for the decomposition of the dual bases.

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1. Introduction

The Schur functions are a fundamental object of study in the areas of algebraic combinatorics, representation theory, and geometry. They were introduced by Cauchy in 1815 [7] and appeared in Schur's seminal dissertation [36] as the characters of the irreducible representations of the general linear group $GL(n,\mathbb{C})$. Schur functions can be generated by means of divided difference operators, raising operators, matrix determinants, and monomial weights. (See texts such as [11,29,35,41] for details.) The multiplication of Schur functions is equivalent to the Schubert calculus on intersections of subspaces of a vector space [39]. The Schur functions form an orthonormal basis for the graded Hopf algebra Sym of symmetric functions [12]. Symmetric functions appear in classical invariant theory results such as the Chevalley–Shephard–Todd Theorem [9, 37] as well as more recent developments such as the theory of Macdonald polynomials [9, 28], nonsymmetric Macdonald polynomials [31], and their related combinatorics [16–18]. The algebra Sym of symmetric functions generalizes to both a nonsymmetric analogue $QSym$ [13] and a noncommutative analogue $NSym$ [14].

Stanley laid the foundation for the algebra $QSym$ of quasisymmetric functions through his work on P-partitions [38]. Gessel [13] formalized the definition of quasisymmetric functions and introduced the fundamental basis. Malvenuto and Reutenauer [30,32] and Ehrenborg [10] further developed the Hopf algebra structure of $QSym$, which is the Hopf algebra dual to the noncommutative symmetric functions $NSym$. $QSym$ also plays an important role in permutation enumeration [15] and reduced decompositions for finite Coxeter groups [40]. Quasisymmetric functions appear in probability theory through the study of random walks [19] and riffle shuffles [42]. They also arise in representation theory as representations of Lie algebras [15], general linear Lie superalgebras [26], and in the study of Hecke algebras [21]. Discrete geometers use quasisymmetric functions in the study of the **cd**-index [5] and as flags in graded posets [10]. Quasisymmetric functions are ubiquitous in combinatorics in part because $QSym$ is the terminal object in the category of combinatorial Hopf algebras [1].

In [22], Haglund et al. introduced a new basis for quasisymmetric functions called the *quasisymmetric Schur functions* $\{\check{\mathscr{S}}_\gamma\}_\gamma$. The quasisymmetric Schur functions are specializations of nonsymmetric Macdonald polynomials obtained by setting $q = t = 0$ in the combinatorial formula described in [20] and summing the resulting Demazure atoms over all weak compositions which collapse to the same strong composition. This new basis satisfies many properties similar to those enjoyed by the Schur functions including a Robinson–Schensted–Knuth style bijection with matrices [23], a Pieri-style multiplication rule [22], and an omega operation [33]. Haglund et al. [23] provide a refinement of the Littlewood–Richardson rule which gives a formula for the coefficients appearing in the product of a quasisymmetric Schur function and a Schur function when expanded in terms of the quasisymmetric Schur function basis. The quasisymmetric Schur functions are generated by fillings of composition diagrams analogously to how Schur functions are generated by semistandard Young tableaux. In representation theory, quasisymmet-