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Anti-powers in infinite words

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A B S T R A C T

In combinatorics of words, a concatenation of k consecutive equal blocks is called a power of order k. In this paper we take a different point of view and define an anti-power of order k as a concatenation of k consecutive pairwise distinct blocks of the same length. As a main result, we show that every infinite word contains powers of any order or anti-powers of any order. That is, the existence of powers or anti-powers is an unavoidable regularity. Indeed, we prove a stronger result, which relates the density of anti-powers to the existence of a factor that occurs with arbitrary exponent. As a consequence, we show that in every aperiodic uniformly recurrent word, anti-powers of every order begin at every position. We further show that every infinite word avoiding anti-powers of order 3 is ultimately periodic, while there exist aperiodic words avoiding anti-powers of order 4. We also show that there exist aperiodic recurrent words avoiding anti-powers of order 6.

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1. Introduction

Ramsey theory is an old and important area of combinatorics. Since the original result of 1930 by F. Ramsey [7] research developed in several directions, but the crux of the matter remains the study of regularities that must arise in large combinatorial structures. These kinds of regularities classically concern substructures formed by *all-equal* elements (e.g., a monochromatic clique in an edge-colored graph).

However, at the beginning of the 70s, Erdős, Simonovits and T. Sós [3] started the study of *anti-Ramsey* theory, that is, the study of regularities that concern *all-distinct* objects (e.g., a subgraph of an edge-colored graph in which all edges have different colors, often called a *rainbow* — see [4] for a survey).

In combinatorics on words, Ramsey theory found applications through some important results stating the existence of unavoidable regularities. Formally, an *unavoidable regularity* is a property P such that it is not possible to construct arbitrarily long words not satisfying P (cf. [1]). Most of the main results about unavoidable regularities in words were originally stated in other areas of combinatorics, e.g., the Ramsey, van der Waerden and Shirshov theorems (see [1,5,6] for further details). All these theorems, however, establish the existence, in every sufficiently long word, of regular substructures. In this paper, we give an anti-Ramsey result in the context of combinatorics on words.

Regularities in words are often associated with *repetitions*, also called *powers*. A power of order k is a concatenation of k identical copies of the same word. The most simple power is a square (a power of order 2). As noted by Thue in 1906 [8], every sufficiently long binary word must contain a square, but there exist arbitrarily long words over a 3-letter alphabet avoiding squares, that is, not containing any square as a block of contiguous letters (in terms of combinatorics on words, a *factor*). This shows that the avoidability of powers depends on the alphabet size.

In this paper we introduce the notion of an anti-power. An anti-power of order k, or simply a *k-anti-power*, is a concatenation of k consecutive pairwise distinct words of the same length. E.g., *aabaaabbbaba* is a 4-anti-power. A simple computation shows that there are in general much more anti-powers than powers for a fixed length and a fixed order; yet there are much less anti-powers than possible factors of the same given length.

Let us consider as an example the well-known Thue–Morse word

$$t = 0110100110010110100101100110100110010110011010 \cdots$$

Starting from $n = 0$, the n-th term is given by the parity of the number of 1s in the binary expansion of n. The Thue–Morse word does not contain *overlaps*, i.e., factors of the form *awawa* for a letter a and a word w. In particular, the Thue–Morse word does not contain 3-powers.

The shortest prefix of the Thue–Morse word that is a 2-antipower is 01. The shortest prefix that is a 3-anti-power is $01101 \cdot 00110 \cdot 01011$, of length 15. One can verify that the shortest 4-anti-power prefix has length 20. The first few lengths of the shortest prefixes