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Reflexive polytopes arising from perfect graphs

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A B S T R A C T

Reflexive polytopes form one of the distinguished classes of lattice polytopes. Especially reflexive polytopes which possess the integer decomposition property are of interest. In the present paper, by virtue of the algebraic technique on Grönbner bases, a new class of reflexive polytopes which possess the integer decomposition property and which arise from perfect graphs will be presented. Furthermore, the Ehrhart δ-polynomials of these polytopes will be studied.

0. Background

The reflexive polytope is one of the keywords belonging to the current trends on the research of convex polytopes. In fact, many authors have studied reflexive polytopes from viewpoints of combinatorics, commutative algebra and algebraic geometry. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related with mirror symmetry (see, e.g., [2,5]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence ([17]) and all of them are known up to dimension 4 ([16]). Moreover, every lattice polytope is a face of a reflexive

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polytope ([7]). We are especially interested in reflexive polytopes with the integer decomposition property, where the integer decomposition property is particularly important in the theory and application of integer programing [23, §22.10]. A lattice polytope which possesses the integer decomposition property is normal and very ample. These properties play important roles in algebraic geometry. Hence to find new classes of reflexive polytopes with the integer decomposition property is one of the most important problem. For example, the following classes of reflexive polytopes with the integer decomposition property are known:

- Centrally symmetric configurations ([20]).
- Reflexive polytopes arising from the order polytopes and the chain polytopes of finite partially ordered sets ([11,12,14,15]).
- Reflexive polytopes arising from the stable sets polytopes of perfect graphs ([21]).

Following the previous work [21] the present paper discusses a new class of reflexive polytopes which possess the integer decomposition property and which arise from perfect graphs.

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1. Perfect graphs and reflexive polytopes

Recall that a *lattice polytope* is a convex polytope all of whose vertices have integer coordinates. We say that a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is *reflexive* if the origin of $\mathbb{R}^d$ belongs to the interior of $\mathcal{P}$ and if the dual polytope

$$\mathcal{P}^\vee = \{\mathbf{x} \in \mathbb{R}^d \,:\, \langle \mathbf{x}, \mathbf{y} \rangle \leq 1, \, \forall \mathbf{y} \in \mathcal{P}\}$$

is again a lattice polytope. Here $\langle \mathbf{x}, \mathbf{y} \rangle$ is the canonical inner product of $\mathbb{R}^d$. A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ possesses the *integer decomposition property* if, for each integer $n \geq 1$ and for each $\mathbf{a} \in n\mathcal{P} \cap \mathbb{Z}^d$, where $n\mathcal{P} = \{n\mathbf{a} \,:\, \mathbf{a} \in \mathcal{P}\}$, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_n$ belonging to $\mathcal{P} \cap \mathbb{Z}^d$ with $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_n$.

Let G be a finite simple graph on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G)$ the set of edges of G. (A finite graph G is called simple if G possesses no loop and no multiple edge.) A subset $W \subset [d]$ is called *stable* if, for all i and j belonging to W with $i \neq j$, one has $\{i, j\} \notin E(G)$. We remark that a stable set is often called an *independent set*. A *clique* of G is a subset $W \subset [d]$ which is a stable set of the complementary graph $\overline{G}$ of G. The *chromatic number* of G is the smallest integer $t \geq 1$ for which there exist stable set $W_1, \ldots, W_t$ of G with $[d] = W_1 \cup \cdots \cup W_t$. A finite simple graph G is said to be *perfect* ([4]) if, for any induced subgraph H of G including G itself, the chromatic number of