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Characteristic polynomials of Linial arrangements for exceptional root systems

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A B S T R A C T

The (extended) Linial arrangement $\mathcal{L}_\Phi^m$ is a certain finite truncation of the affine Weyl arrangement of a root system Φ with a parameter m. Postnikov and Stanley conjectured that all roots of the characteristic polynomial of $\mathcal{L}_\Phi^m$ have the same real part, and this has been proved for the root systems of classical types.

In this paper we prove that the conjecture is true for exceptional root systems when the parameter m is sufficiently large.

The proof is based on representations of the characteristic quasi-polynomials in terms of Eulerian polynomials.

Contents

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1. Introduction

1.1. Background

A hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is a finite collection of affine hyperplanes in an ℓ-dimensional vector space $\mathbb{K}^\ell$. Despite its simplicity, the theory of hyperplane arrangements has fruitful connections with many areas in mathematics ([20,23]). One of the most important invariants of an arrangement $\mathcal{A}$ is the *characteristic polynomial* $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$. Indeed the characteristic polynomial is related to several other invariants, such as the Poincaré polynomial of the complexified complement $M(\mathcal{A})$ [19], the number of chambers for real arrangements [31], the number of $\mathbb{F}_q$-rational points [10,26], Chern classes of certain vector bundles [18,1], and lattice points countings [7,14–16,30].

1.2. Main results

Let $V = \mathbb{R}^\ell$ be an ℓ-dimensional Euclidean space. Let $\Phi \subset V^*$ be an irreducible root system. Fix a positive system $\Phi^+ \subset \Phi$. For a positive root $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, define

$$H_{\alpha,k} = \{x \in V \mid \alpha(x) = k\}.$$

The set of all such hyperplanes is called the affine Weyl arrangement. Finite truncations of the affine Weyl arrangement have received considerable attention ([2–5,11,21,22,27,29]). Among others, the (extended) Linial arrangement $\mathcal{L}_\Phi^m$ is defined by

$$\mathcal{L}_\Phi^m = \{H_{\alpha,k} \mid \alpha \in \Phi^+, k = 1, 2, \dots, m\},$$

(where $\mathcal{L}_\Phi^0 = \emptyset$ by convention). In [21], Postnikov and Stanley studied combinatorial aspects of Linial arrangements. They posed the following conjecture.

Conjecture 1.1 *([21, Conjecture 9.14]). Suppose $m \geq 1$. Then every root $\alpha \in \mathbb{C}$ of the equation $\chi(\mathcal{L}_\Phi^m, t) = 0$ satisfies $\operatorname{Re}\alpha = \frac{mh}{2}$, where h denotes the Coxeter number of Φ.*