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Products of abstract polytopes

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ABSTRACT

Given two convex polytopes, the join, the cartesian product and the direct sum of them are well understood. In this paper we extend these three kinds of products to abstract polytopes and introduce a new product, called the topological product, which also arises in a natural way. We show that these products have unique prime factorization theorems. We use this to compute the automorphism group of a product in terms of the automorphism groups of the factors and show that (non trivial) products are almost never regular or two-orbit polytopes. We finish the paper by studying the monodromy group of a product, show that such a group is always an extension of a symmetric group, and give some examples in which this extension splits.

1. Introduction

In school we all dealt, in one way or another, with solids such as prisms and pyramids, but maybe also with bipyramids. The aim of this paper is to generalize these solids as different products of abstract polytopes, and study their symmetry and combinatorial properties.

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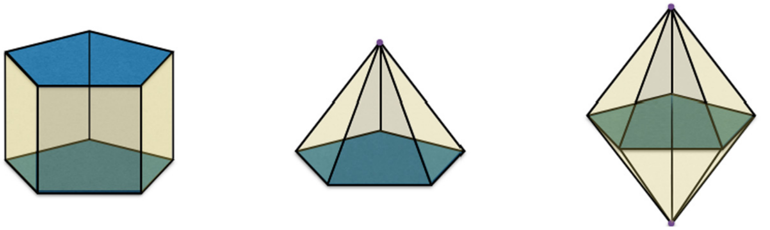

Fig. 1. A prism, pyramid and bipyramid over a pentagon.

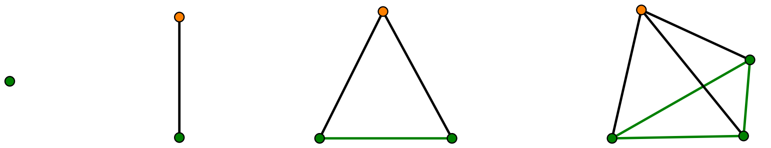

Fig. 2. A d-simplex is the join product of a point and a $(d-1)$-simplex.

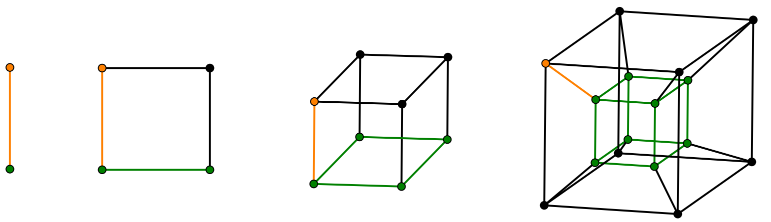

Fig. 3. A d-cube is the cartesian product of an edge and a $(d-1)$-cube.

Prisms, pyramids and bipyramids over polygons (see Fig. 1) can be seen as a product of a polygon with either a segment or a point. However, these are three different kinds of products. While prisms are the cartesian product of a segment with a polygon, pyramids are the join product of a point with a polygon and bipyramids are the direct product of a segment with a polygon. In the theory of convex polytopes the generalization of these three notions are the cartesian product, the join product and the direct sum, respectively ([7]). Given two full-dimensional convex polytopes $\mathcal{P} \subset \mathbb{R}^n$ and $\mathcal{Q} \subset \mathbb{R}^m$, their products are defined as follows.

The join of $\mathcal{P}$ and $\mathcal{Q}$ (denoted by $\mathcal{P} \bowtie \mathcal{Q}$) is obtained by embedding $\mathcal{P}$ and $\mathcal{Q}$ in skew affine subspaces of $\mathbb{R}^{n+m+1}$ and taking the convex hull of their vertices. For example, for each $d \geq 1$, a d-simplex can be seen as the join of a point and a $(d-1)$-simplex (Fig. 2).

The cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ (denoted $\mathcal{P} \times \mathcal{Q}$) is obtained by taking the convex hull of $V(\mathcal{P}) \times V(\mathcal{Q})$ in $\mathbb{R}^{n+m}$. The classical example in this case, is to see a d-cube as the cartesian product of an edge – or line segment – with a $(d-1)$-cube (as in Fig. 3).

The direct sum of $\mathcal{P}$ and $\mathcal{Q}$ (or free sum, denoted by $\mathcal{P} \oplus \mathcal{Q}$) is slightly more complicated to state. We first require that $\mathcal{P}$ and $\mathcal{Q}$ contain in their relative interiors the origins of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then the direct sum is the convex hull of all the points of the