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r-Stable hypersimplices ☆

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ABSTRACT

Hypersimplices are well-studied objects in combinatorics, optimization, and representation theory. For each hypersimplex, we define a new family of subpolytopes, called r-stable hypersimplices, and show that a well-known regular unimodular triangulation of the hypersimplex restricts to a triangulation of each r-stable hypersimplex. For the case of the second hypersimplex defined by the two-element subsets of an n-set, we provide a shelling of this triangulation that sequentially shells each r-stable sub-hypersimplex. In this case, we utilize the shelling to compute the Ehrhart h^*-polynomials of these polytopes, and the hypersimplex, via independence polynomials of graphs. For one such r-stable hypersimplex, this computation yields a connection to CR mappings of Lens spaces via Ehrhart–MacDonald reciprocity.

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1. Introduction

1.1. Hypersimplices and Ehrhart theory

Fix integers $0 < k < n$ and let $[n] := \{1, 2, \dots, n\}$. The **characteristic vector** of a k-subset I of $[n]$ is the $(0,1)$-vector $\epsilon_I := (\epsilon_1, \dots, \epsilon_n)$ such that $\epsilon_i = 1$ for $i \in I$ and $\epsilon_i = 0$ for $i \notin I$. The (n,k)**-hypersimplex**, denoted $\Delta_{n,k}$, is the $(n-1)$-dimensional polytope that is the convex hull of the characteristic vectors of all k-subsets of $[n]$. Hypersimplices appear naturally in algebraic and geometric contexts, as well as in pure and applied combinatorial contexts. In [19], Stanley proved that the normalized volume of the (n,k)-hypersimplex is the Eulerian number $A_{k,n-1}$. De Loera, Sturmfels, and Thomas studied the connection between triangulations of the hypersimplex and Gröbner bases via toric algebra [6]. In [14], Lam and Postnikov showed four useful triangulations of the hypersimplex are all identical. In [12], Katzman gave an algebraic description of the Ehrhart h^*-polynomial of the hypersimplex, and in [17], Li gave a second interpretation in terms of exceedances and descents. These investigations have proven fruitful for our understanding of hypersimplices, but many interesting questions remain unanswered.

Challenging open problems pertaining to hypersimplices lie in the field of Ehrhart theory. A **lattice polytope** $\mathcal{P} \subset \mathbb{R}^n$ of dimension d is the convex hull of finitely many points in $\mathbb{Z}^n$ that together affinely span a d-dimensional hyperplane. For $t \in \mathbb{Z}_{>0}$, set $t\mathcal{P} := \{tp : p \in \mathcal{P}\}$, and let $L_\mathcal{P}(t) = |\mathbb{Z}^n \cap t\mathcal{P}|$. In [7], Ehrhart proved that with the polynomial basis $\{\binom{t+d-i}{d} : i \in [0,d] \cap \mathbb{Z}\}$, for any lattice polytope $\mathcal{P}$ we have

$$L_\mathcal{P}(t) = \sum_{i=0}^{d} h_i^* \binom{t+d-i}{d}.$$

The polynomial $L_\mathcal{P}(t)$ is called the **Ehrhart polynomial of** $\mathcal{P}$ and has connections to commutative algebra, algebraic geometry, combinatorics, and discrete and convex geometry. In [20], Stanley proved that $h_i^* \in \mathbb{Z}_{\geq 0}$ for all i. We call the polynomial generating function for these coefficients $h^*(\mathcal{P}; z) := h_0^* + h_1^* x + \cdots + h_d^* x^d$ the h^***-polynomial** (or δ**-polynomial**) of $\mathcal{P}$. Various properties of $\mathcal{P}$ are reflected in its h^*-polynomial. For example, $\mathrm{vol}(\mathcal{P}) = \frac{\sum_i h_i^*}{d!}$, where $\mathrm{vol}(\mathcal{P})$ denotes the Euclidean volume (Lebesgue measure) of $\mathcal{P}$ with respect to the integer lattice contained in the hyperplane spanned by $\mathcal{P}$. Another sought-after (but not always achieved) property of $h^*(\mathcal{P}; x)$ is *unimodality*. A polynomial $a_0 + \cdots + a_d x^d$ is called **unimodal** if there exists an index j, $0 \leq j \leq d$, such that $a_{i-1} \leq a_i$ for $i \leq j$, and $a_i \geq a_{i+1}$ for $i \geq j$. Unimodality of h^*-polynomials is an area of active research [10,12,18]. In the special case of the hypersimplex, Haws, De Loera, and Köppe computationally verified that the h^*-polynomial of $\Delta_{n,k}$ is unimodal for $n \leq 75$ and conjectured that this holds in general [5].