Journal of Combinatorial Theory, Series A 156 (2018) 44–60

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Non-trivially intersecting multi-part families

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ARTICLE INFO

Article history:
Received 29 March 2017
Available online xxxx

Keywords:
Extremal set theory
Intersecting families
Erdős–Ko–Rado theorem
Hilton–Milner theorem

ABSTRACT

We say a family of sets is *intersecting* if any two of its sets intersect, and we say it is *trivially intersecting* if there is an element which appears in every set of the family. In this paper we study the maximum size of a non-trivially intersecting family in a natural "multi-part" setting. Here the ground set is divided into parts, and one considers families of sets whose intersection with each part is of a prescribed size. Our work is motivated by classical results in the single-part setting due to Erdős, Ko and Rado, and Hilton and Milner, and by a theorem of Frankl concerning intersecting families in this multi-part setting. In the case where the part sizes are sufficiently large we determine the maximum size of a non-trivially intersecting multi-part family, disproving a conjecture of Alon and Katona.

1. Introduction

We say that a family of sets $\mathcal{F}$ is *intersecting* if the intersection of any two of its sets is non-empty. Moreover, we say that $\mathcal{F}$ is *trivially intersecting* if there is an element i such that $i \in F$ for each set $F \in \mathcal{F}$. The Erdős–Ko–Rado theorem [9] says that if $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family of k-element subsets of an n-element ground set, and if $1 \leq k \leq n/2$, then

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https://doi.org/10.1016/j.jcta.2017.12.001
0097-3165/

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

This bound is sharp: it is attained, for example, by the trivially intersecting family $\mathcal{F}^{\mathrm{EKR}}(n,k)$ consisting of all k-element subsets of $[n]$ which contain 1. In fact, for $k < n/2$ this is essentially the only extremal family. We remark that if $k > n/2$ then $\binom{[n]}{k}$ itself is intersecting.

The Erdős–Ko–Rado theorem is of fundamental importance in extremal set theory, and many related questions have been asked and answered. In particular, Hilton and Milner [17] proved a stability version of the Erdős–Ko–Rado theorem, showing that for $2 \leq k < n/2$, the maximum size of a non-trivially intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ is

$$M^{\mathrm{HM}}(n,k) := \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

This bound is sharp: it is attained, for example, by the family $\mathcal{F}^{\mathrm{HM}}(n,k)$ consisting of the set $F = \{2,\dots,k+1\}$, in addition to all possible sets that contain 1 and intersect F. Note that this family is significantly smaller than the Erdős–Ko–Rado bound. In particular, for constant k and $n \to \infty$, we have $\left|\mathcal{F}^{\mathrm{HM}}(n,k)\right| = o\left(\left|\mathcal{F}^{\mathrm{EKR}}(n,k)\right|\right)$. We remark that if $k = 1$ then every intersecting family is trivially intersecting.

1.1. Multi-part intersecting families

A natural "multi-part" extension of the Erdős–Ko–Rado problem was introduced by Frankl [12], in connection with a result of Sali [20] (see also [13]). For $p \geq 1$ and $n_1,\dots,n_p \geq 1$, our ground set is $[\sum_s n_s] = \{1,2,\dots,\sum_s n_s\}$. We interpret this ground set as the disjoint union of p parts $[n_1],\dots,[n_p]$ and we write $[\sum_s n_s] = \bigsqcup_s [n_s]$. More generally, for sets $F_1 \in 2^{[n_1]},\dots,F_p \in 2^{[n_p]}$ let $\bigsqcup_s F_s$ be the subset of $\bigsqcup_s [n_s]$ with F_s in part s, and for families $\mathcal{F}_1 \subseteq 2^{[n_1]},\dots,\mathcal{F}_p \subseteq 2^{[n_p]}$ let $\prod_s \mathcal{F}_s = \{\bigsqcup_s F_s : F_s \in \mathcal{F}_s\}$. Consider $k_1 \in [n_1],\dots k_p \in [n_p]$, so that $\prod_s \binom{[n_s]}{k_s}$ is the collection of all subsets of $\bigsqcup_s [n_s]$ which have exactly k_s elements in each part s. Families of the form $\mathcal{F} \subseteq \prod_s \binom{[n_s]}{k_s}$ are the natural generalization of k-uniform families to the multi-part setting. Note that a multi-part family is intersecting if any two of its sets intersect in at least one of the parts.

Frankl proved that for any $p \geq 1$, any $n_1,\dots,n_p$ and any $k_1,\dots,k_p$ satisfying $1 \leq k_s \leq n_s/2$, the maximum size of a multi-part intersecting family $\mathcal{F} \subseteq \prod_s \binom{[n_s]}{k_s}$ is

$$\max_{t\in[p]} \binom{n_t-1}{k_t-1} \prod_{s\neq t} \binom{n_s}{k_s} = \left(\max_{t\in[p]} \frac{k_t}{n_t}\right) \prod_{s=1}^{p} \binom{n_s}{k_s}.$$

This bound is sharp: it is attained, for example, by a product family of the form

$$\binom{[n_1]}{k_1} \times \cdots \times \binom{[n_{t-1}]}{k_{t-1}} \times \mathcal{F}^{\mathrm{EKR}}(n_t,k_t) \times \binom{[n_{t+1}]}{k_{t+1}} \times \cdots \times \binom{[n_p]}{k_p}.$$