Journal of Combinatorial Theory, Series A 157 (2018) 402–426

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

# Classes and equivalence of linear sets in $\mathrm{PG}(1, q^n)$ ☆

Bence Csajbók [a,b], Giuseppe Marino [b], Olga Polverino [b]

[a] *MTA–ELTE Geometric and Algebraic Combinatorics Research Group, ELTE Eötvös Loránd University, Budapest, Hungary, Department of Geometry, 1117 Budapest, Pázmány P. stny. 1/C, Hungary*

[b] *Dipartimento di Matematica e Fisica, Università degli Studi della Campania "Luigi Vanvitelli", Viale Lincoln 5, I-81100 Caserta, Italy*

ARTICLE INFO

*Article history:*
Received 12 April 2016
Available online xxxx

*Keywords:*
Linearized polynomial
Linear set
Blocking set
MRD-code

ABSTRACT

The equivalence problem of $\mathbb{F}_q$-linear sets of rank $n$ of $\mathrm{PG}(1, q^n)$ is investigated, also in terms of the associated variety, projecting configurations, $\mathbb{F}_q$-linear blocking sets of Rédei type and MRD-codes. We call an $\mathbb{F}_q$-linear set $L_U$ of rank $n$ in $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ *simple* if for any $n$-dimensional $\mathbb{F}_q$-subspace $V$ of $W$, $L_V$ is $\mathrm{P\Gamma L}(2, q^n)$-equivalent to $L_U$ only when $U$ and $V$ lie on the same orbit of $\Gamma\mathrm{L}(2, q^n)$. We prove that $U = \{(x, \mathrm{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}\}$ defines a simple $\mathbb{F}_q$-linear set for each $n$. We provide examples of non-simple linear sets not of pseudoregulus type for $n > 4$ and we prove that all $\mathbb{F}_q$-linear sets of rank 4 are simple in $\mathrm{PG}(1, q^4)$.



☆ The research was supported by Ministry of Education, University and Research of Italy MIUR (Project PRIN 2012 "Geometrie di Galois e strutture di incidenza") and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA – INdAM). The first author was partially supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by OTKA Grant No. K 124950.

*E-mail addresses:* csajbokb@cs.elte.hu (B. Csajbók), giuseppe.marino@unicampania.it (G. Marino), olga.polverino@unicampania.it (O. Polverino).

https://doi.org/10.1016/j.jcta.2018.03.007

## 1. Introduction

Linear sets are natural generalizations of subgeometries. Let $\Lambda = \mathrm{PG}(W, \mathbb{F}_{q^n}) = PG(r-1, q^n)$, where $W$ is a vector space of dimension $r$ over $\mathbb{F}_{q^n}$. A point set $L$ of $\Lambda$ is said to be an $\mathbb{F}_q$-*linear set* of $\Lambda$ of rank $k$ if it is defined by the non-zero vectors of a $k$-dimensional $\mathbb{F}_q$-vector subspace $U$ of $W$, i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

The maximum field of linearity of an $\mathbb{F}_q$-linear set $L_U$ is $\mathbb{F}_{q^t}$ if $t \mid n$ is the largest integer such that $L_U$ is an $\mathbb{F}_{q^t}$-linear set. In the recent years, starting from the paper [21] by Lunardon, linear sets have been used to construct or characterize various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [28], see also [17].

One of the most natural questions about linear sets is their equivalence. Two linear sets $L_U$ and $L_V$ of $\mathrm{PG}(r-1, q^n)$ are said to be $\mathrm{P\Gamma L}$-*equivalent* (or simply *equivalent*) if there is an element $\varphi$ in $\mathrm{P\Gamma L}(r, q^n)$ such that $L_U^{\varphi} = L_V$. In the applications it is crucial to have methods to decide whether two linear sets are equivalent or not. For $f \in \Gamma\mathrm{L}(r, q^n)$ we have $L_{U^f} = L_U^{\varphi_f}$, where $\varphi_f$ denotes the collineation of $\mathrm{PG}(W, \mathbb{F}_{q^n})$ induced by $f$. It follows that if $U$ and $V$ are $\mathbb{F}_q$-subspaces of $W$ belonging to the same orbit of $\Gamma\mathrm{L}(r, q^n)$, then $L_U$ and $L_V$ are equivalent. The above condition is only sufficient but not necessary to obtain equivalent linear sets. This follows also from the fact that $\mathbb{F}_q$-subspaces of $W$ with different ranks can define the same linear set, for example $\mathbb{F}_q$-linear sets of $\mathrm{PG}(r-1, q^n)$ of rank $k \geq rn - n + 1$ are all the same: they coincide with $\mathrm{PG}(r-1, q^n)$. As it was showed recently in [8], if $r = 2$, then there exist $\mathbb{F}_q$-subspaces of $W$ of the same rank $n$ but on different orbits of $\Gamma\mathrm{L}(2, q^n)$ defining the same linear set of $\mathrm{PG}(1, q^n)$.

This observation motivates the following definition. An $\mathbb{F}_q$-linear set $L_U$ of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$ with maximum field of linearity $\mathbb{F}_q$ is called *simple* if for each $\mathbb{F}_q$-subspace $V$ of $W$, $L_U = L_V$ only if $U$ and $V$ are in the same orbit of $\Gamma\mathrm{L}(r, q^n)$ or, equivalently, if for each $\mathbb{F}_q$-subspace $V$ of $W$, $L_V$ is $\mathrm{P\Gamma L}(r, q^n)$-equivalent to $L_U$ only if $U$ and $V$ are in the same orbit of $\Gamma\mathrm{L}(r, q^n)$.

Natural examples of simple linear sets are the subgeometries (cf. [20, Theorem 2.6] and [16, Section 25.5]). In [6] it was proved that $\mathbb{F}_q$-linear sets of rank $n+1$ of $\mathrm{PG}(2, q^n)$ admitting $(q+1)$-secants are simple. This allowed the authors to translate the question of equivalence to the study of the orbits of the stabilizer of a subgeometry on subspaces and hence to obtain the complete classification of $\mathbb{F}_q$-linear blocking sets in $\mathrm{PG}(2, q^4)$. Until now, the only known examples of non-simple linear sets are those of pseudoregulus type of $\mathrm{PG}(1, q^n)$ for $n \geq 5$ and $n \neq 6$, see [8].

In this paper we focus on linear sets of rank $n$ of $\mathrm{PG}(1, q^n)$. We first introduce a method which can be used to find non-simple linear sets of rank $n$ of $\mathrm{PG}(1, q^n)$. Let $L_U$ be a