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Extremal bounds for bootstrap percolation in the hypercube

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ABSTRACT

The *r-neighbour bootstrap percolation process* on a graph G starts with an initial set A_0 of "infected" vertices and, at each step of the process, a healthy vertex becomes infected if it has at least r infected neighbours (once a vertex becomes infected, it remains infected forever). If every vertex of G eventually becomes infected, then we say that A_0 *percolates*.

We prove a conjecture of Balogh and Bollobás which says that, for fixed r and $d \to \infty$, every percolating set in the d-dimensional hypercube has cardinality at least $\frac{1+o(1)}{r}\binom{d}{r-1}$. We also prove an analogous result for multidimensional rectangular grids. Our proofs exploit a connection between bootstrap percolation and a related process, known as *weak saturation*. In addition, we improve on the best known upper bound for the minimum size of a percolating set in the hypercube. In particular, when $r = 3$, we prove that the minimum cardinality of a percolating set in the d-dimensional hypercube is $\left\lceil \frac{d(d+3)}{6} \right\rceil + 1$ for all $d \geq 3$.

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1. Introduction

Given a positive integer r and a graph G, the *r-neighbour bootstrap percolation process* begins with an initial set of "infected" vertices of G and, at each step of the process, a vertex becomes infected if it has at least r infected neighbours. More formally, if A_0 is the initial set of infected vertices, then the set of vertices that are infected after the jth step of the process for $j \geq 1$ is defined by

$$A_j := A_{j-1} \cup \{v \in V(G) : |N_G(v) \cap A_{j-1}| \geq r\},$$

where $N_G(v)$ denotes the neighbourhood of v in G. We say that A_0 *percolates* if $\bigcup_{j=0}^{\infty} A_j = V(G)$. Bootstrap percolation was introduced by Chalupa, Leath and Reich [14] as a mathematical simplification of existing dynamic models of ferromagnetism, but it has also found applications in the study of other physical phenomena such as crack formation and hydrogen mixtures (see Adler and Lev [1]). In addition, advances in bootstrap percolation have been highly influential in the study of more complex processes including, for example, the Glauber dynamics of the Ising model [22].

The main extremal problem in bootstrap percolation is to determine the minimum cardinality of a set which percolates under the r-neighbour bootstrap percolation process on G; we denote this by $m(G, r)$. An important case is when G is the *d-dimensional hypercube* Q_d; i.e., the graph with vertex set $\{0, 1\}^d$ in which two vertices are adjacent if they differ in exactly one coordinate. Balogh and Bollobás [4] (see also [8,9]) made the following conjecture.

Conjecture 1.1 *(Balogh and Bollobás [4]). For fixed $r \geq 3$ and $d \to \infty$,*

$$m(Q_d, r) = \frac{1 + o(1)}{r} \binom{d}{r-1}.$$

The upper bound of Conjecture 1.1 is not difficult to prove. Simply let A_0 consist of all vertices on "level $r - 2$" of Q_d and an approximate Steiner system on level r, whose existence is guaranteed by an important theorem of Rödl [27]; see Balogh, Bollobás and Morris [8] for more details. Note that, under certain conditions on d and r, the approximate Steiner system in this construction can be replaced with an exact Steiner system (using, for example, the celebrated result of Keevash [20]). In this special case, the percolating set has cardinality $\frac{1}{r}\binom{d}{r-1} + \binom{d}{r-2}$ which yields

$$m(Q_d, r) \leq \frac{d^{r-1}}{r!} + \frac{d^{r-2}(r+2)}{2r(r-2)!} + O\left(d^{r-3}\right). \tag{1.2}$$

Lower bounds have been far more elusive; previously, the best known lower bound on $m(Q_d, r)$ for fixed $r \geq 3$ was only linear in d [8]. In this paper, we prove Conjecture 1.1.

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