Journal of Combinatorial Theory, Series A 157 (2018) 427–434

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

New inequalities for families without k pairwise disjoint members

Peter Frankl [a], Andrey Kupavskii [b,c,1]

[a] Rényi Institute, Budapest, Hungary
[b] Moscow Institute of Physics and Technology, Russian Federation
[c] University of Birmingham, United Kingdom

A R T I C L E I N F O

Article history:
Received 12 June 2017
Available online 15 March 2018

Keywords:
Families with no matchings
Kleitman's problem
Cross-dependent families
Extremal set theory

A B S T R A C T

Some best possible inequalities are established for k-*partition-free* families (cf. Definition 1) and they are applied to prove a sharpening of a classical result of Kleitman concerning families without k pairwise disjoint members.

1. Introduction

Let n be a positive integer, $[n] = \{1, 2, \ldots, n\}$ is the standard n-element set, $2^{[n]}$ its power set. For an integer $k \geq 2$ a family $\mathcal{F} \subset 2^{[n]}$ is called k-*dependent* if it contains no k pairwise disjoint members. Similarly, if $\mathcal{F}_1, \ldots, \mathcal{F}_k \subset 2^{[n]}$ are not necessarily distinct families, we say that they are *cross-dependent* if there is no choice of $F_i \in \mathcal{F}_i$, $i = 1, \ldots, k$, such that $F_1, \ldots, F_k$ are pairwise disjoint.

An important classical result of Kleitman [5] determines the maximal size, $|\mathcal{F}|$ of a k-dependent family $\mathcal{F} \subset 2^{[n]}$ for the cases $n \equiv -1$ or $0 \pmod{k}$. In a recent paper [2], Kupavskii and the author determined the maximum of $|\mathcal{F}_1| + \ldots + |\mathcal{F}_k|$ for cross-dependent

E-mail addresses: peter.frankl@gmail.com (P. Frankl), kupavskii@yandex.ru (A. Kupavskii).

[1] The research of A. Kupavskii was supported in part by the RNF 16-11-10014.

https://doi.org/10.1016/j.jcta.2018.01.007
0097-3165/

families $\mathcal{F}_i$ for all values of $n \geq k \geq 3$. (Let us note that the easy case of $k = 2$ was already solved by Erdős, Ko and Rado [1].)

Definition 1. For $k \geq 3$ and a family $\mathcal{F} \subset 2^{[n]}$ we say that $\mathcal{F}$ is *k-partition-free* if $\mathcal{F}$ contains no k pairwise disjoint members whose union is $[n]$.

Being *k-partition-free* is slightly less restrictive than being k-dependent.

For $0 \leq j \leq n$ let us use the notations $\mathcal{F}^{(j)} = \mathcal{F} \cap \binom{[n]}{j}$, $f^{(j)} = |\mathcal{F}^{(j)}|$.

The following inequality is an important discovery of Kleitman [5].

Kleitman Lemma. *Let $\mathcal{F} \subset 2^{[n]}$ be k-partition-free and let $j_1, j_2, \ldots, j_k$ be non-negative integers satisfying $j_1 + \ldots + j_k = n$. Then*

$$\sum_{1 \leq i \leq k} \frac{f^{(j_i)}}{\binom{n}{j_i}} \leq k - 1. \tag{1}$$

The proof of (1) is an easy averaging over all choices of pairwise disjoint sets $G_1, \ldots, G_k$ satisfying $|G_i| = j_i$ and noting that at least one of the relations $G_i \in \mathcal{F}$ fails.

Since the relation $j_1 + \ldots + j_k = n$ is essential for proving (1) it is rather surprising that in certain cases one can prove the analogous inequality even if $j_1 + \ldots + j_k > n$.

Let us first state our inequality for the case $k = 3$.

Theorem 2. *Let $m > \ell > 0$ be integers, $n = 3m - \ell$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is 3-partition-free. Then*

$$\frac{|\mathcal{F}^{(m-\ell)}|}{\binom{n}{m-\ell}} + \frac{|\mathcal{F}^{(m)}|}{\binom{n}{m}} + \frac{|\mathcal{F}^{(m+\ell)}|}{\binom{n}{m+\ell}} \leq 2. \tag{2}$$

Looking at the family $\binom{[n]}{m} \cup \binom{[n]}{m+\ell}$ shows that (2) is best possible.

To state our most general result let us say that the families $\mathcal{F}_1, \ldots, \mathcal{F}_k \subset 2^{[n]}$ are *cross-partition-free* if there is no choice of $F_i \in \mathcal{F}_i$, $i = 1, \ldots, k$ such that $F_1, \ldots, F_k$ form a partition of $[n]$.

Theorem 3. *Let $m > \ell > 0$ be integers, $n = km - \ell$, $k \geq 3$. For $1 \leq i \leq k$ let $\mathcal{F}_i \subset \binom{[n]}{m-\ell} \cup \binom{[n]}{m} \cup \binom{[n]}{m+\ell}$ and suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are cross-partition-free. Then*

$$\sum_{1 \leq i \leq k} \frac{|\mathcal{F}_i^{(m-\ell)}|}{\binom{n}{m-\ell}} + \frac{|\mathcal{F}_i^{(m)}|}{\binom{n}{m}} + (k-2)\frac{|\mathcal{F}_i^{(m+\ell)}|}{\binom{n}{m+\ell}} \leq (k-1)k. \tag{3}$$

Note that for $k = 3$ and $\mathcal{F}_1 = \ldots = \mathcal{F}_k$ the inequality (3) implies (2). The reason that we treat it separately is that both the statement and the proof are simple and hopefully give the reader the motivation to go through the more technical result (3).

The proofs of (2) and (3) are based on Katona's cyclic permutation method (cf. [3,4]).