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Kohnert tableaux and a lifting of quasi-Schur functions

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We introduce the quasi-key basis of the polynomial ring which contains the quasi-Schur polynomials of Haglund, Luoto, Mason and van Willigenburg. We prove that stable limits of quasi-key polynomials are quasi-Schur functions, thus lifting the quasi-Schur basis of quasisymmetric polynomials to the full polynomial ring. The new tool we introduce for this purpose is the combinatorial model of Kohnert tableaux. We use this model to prove that key polynomials expand positively in quasi-key polynomials which in turn expand positively in fundamental slide polynomials introduced earlier by the authors. We give simple combinatorial formulas for these expansions in terms of Kohnert tableaux, lifting the parallel expansions of a Schur function into quasi-Schur functions into fundamental quasisymmetric functions.

1. Introduction

Lascoux and Schützenberger [9] studied a basis for polynomials, called *key polynomials*, that are a lifting of the Schur functions from the ring of symmetric functions to the full polynomial ring. This parallels Macdonald's [11] result that Schubert polynomials

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$$
\begin{array}{ccccccc}
\mathfrak{S}_w & \xrightarrow{\text{nonneg}} & \kappa_a & \xrightarrow{\text{nonneg}} & \mathfrak{Q}_b & \xrightarrow{\text{nonneg}} & \mathfrak{F}_c \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S_w & \xrightarrow{\text{nonneg}} & s_{\mathrm{sort}(a)} & \xrightarrow{\text{nonneg}} & QS_{\mathrm{flat}(b)} & \xrightarrow{\text{nonneg}} & F_{\mathrm{flat}(c)}
\end{array}
$$

Fig. 1. An illustration containing our main results: a right arrow from f to g indicates that f expands nonnegatively into the basis $\{g\}$, and a down arrow from f to g indicates that f stabilizes to g.

are a lifting of Stanley symmetric functions [15]. This raises the more general question: how does one pull back symmetric functions to the polynomial ring? Moreover, can it be done in such a way that properties relating bases in symmetric functions still hold in the polynomial ring? The answer, perhaps, is to broaden the question to include not only symmetric functions, but quasisymmetric functions as well. For an illustration of known and new pull backs, see Fig. 1.

In earlier work [1], the authors introduced the *fundamental slide polynomials* which are pull backs of the fundamental quasisymmetric functions of Gessel [5]. In this paper, we make use of the fundamental slide polynomials to lift another important basis for quasisymmetric functions to the polynomial ring, namely the quasisymmetric Schur functions, which we call *quasi-Schur functions*, of Haglund, Luoto, Mason and van Willigenburg [6]. The quasi-Schur functions have many interesting applications to symmetric functions, quasisymmetric functions and polynomials [3,7,10].

The combinatorial model we introduce for this purpose is based on Kohnert's [8] simple algorithmic model for key polynomials in terms of diagrams. We label the cells of a Kohnert diagram with positive integers in a unique way, creating *Kohnert tableaux*. In essence, these labelings keep track of how cells move under Kohnert's algorithm. In particular, when multiple paths in Kohnert's algorithm yield the same diagram, the labeling gives a canonical choice among these paths.

While the labeling itself does not affect the monomial associated to a Kohnert diagram, Kohnert tableaux have several advantages: for example, they permit a static description of all Kohnert diagrams associated to a given key polynomial, independently of Kohnert's algorithm. Most importantly for us, we need the refined information encoded by this labeling to give definitions that are central for our main results. We define a condition, called quasi-Yamanouchi, on Kohnert tableaux which gives a compact, positive expansion for key polynomials into the fundamental slide basis, similar to using standard Young tableaux in place of semi-standard Young tableaux for Schur functions (the former is a reasonably small finite set while the latter is infinite). Moreover, by imposing an additional restriction on Kohnert tableaux, we partition the terms in the fundamental slide expansion of a key polynomial to form an intermediate basis that we call the *quasi-key polynomials*.

The fundamental slide expansion of quasi-key polynomials is indexed by quasi-Yamanouchi quasi-Kohnert tableaux, and the fundamental quasisymmetric expansion of quasi-Schur polynomials is indexed by standard composition tableaux. We give a simple bijection between these two families of tableaux to prove that quasi-key polynomials stabilize to quasi-Schur functions. In this way, our formula for the expansion of a quasi-