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A topos associated with a colored category

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We show that a functor category whose domain is a colored category is a topos. The topos structure enables us to introduce cohomology of colored categories including quasi-schemoids. If the given colored category arises from an association scheme, then the cohomology coincides with the group cohomology of the factor scheme by the thin residue. Moreover, it is shown that the cohomology of a colored category relates to the standard representation of an association scheme via the Leray spectral sequence.

1. Introduction

Quasi-schemoids have been introduced in [11] generalizing the notion of an *association scheme* [1,18,22] from a small-categorical point view. In a nutshell, the new object is a small category whose morphisms are colored with appropriate combinatorial data. Strong homotopy and representation theory for quasi-schemoids are developed in [10] and [12], respectively.

Once neglecting the combinatorial data in a quasi-schemoid, we have a category with colored morphisms. In what follows, such a category is called a *colored category*. The

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main theorem (Theorem 2.7) in this article enables one to give a topos structure to a functor category whose domain is a colored category and whose objects are functors to the category of sets preserving colors; see Section 2 for the precise definition of the functor category. In consequence, appealing to the topos structure, we define cohomology of a colored category; see Definition 2.8. We have the inclusion functor from the functor category mentioned above to the usual functor category of the underlying category of the colored one. Theorem 2.7 also asserts that the inclusion gives rise to a geometric morphism of topoi whose direct image functor seems to be the *sheafification*.

Applying the cohomology functor to an association scheme, we obtain the group cohomology of the factor scheme by the thin residue; see Proposition 2.13. Then, one might think that the cohomology is not novel for association schemes. However, our attempt to introduce cohomology of colored categories is thought of as the first step to study various cohomologies for such objects encompassing quasi-schemoids and hence association schemes; see Remark 5.7 (ii).

A morphism between colored categories gives rise to a geometric morphism between the topoi associated with the colored categories. Thus the Leray spectral sequence in topos theory may allow us to investigate cohomology of a colored category. In particular, we apply the spectral sequence for considering cohomology of a colored poset; see Example 4.2 below. Moreover, adjoint functors induced by a morphism from an association scheme (X, S_X) to a colored category $(\mathcal{C}, S)$ connect the functor category of $(\mathcal{C}, S)$ with the module category over the Bose–Mesner algebra of (X, S_X). Thus, for example, the cohomology of a colored category relates to the standard representation of an association scheme via the Leray spectral sequence; see Theorem 5.2.

The article is organized as follows. In Section 2, we describe our main theorem. Then *orthodox* cohomology of a colored category is defined. In Section 3, we prove the main theorem. In Section 4, geometric morphisms are investigated in our framework. An application and computational examples of cohomology of colored categories are also described. Section 5 considers the relationship mentioned above between cohomology of a colored category and the standard representation of an association scheme. In the end of the section, observations and expectations for our work are described.

2. Main results

We begin by recalling the definition of a quasi-schemoid. A quasi-schemoid will be referred to as a schemoid in this article. Let $\mathcal{C}$ be a small category and S a partition of the set $mor(\mathcal{C})$ of all morphisms in $\mathcal{C}$; that is, $mor(\mathcal{C}) = \coprod_{\sigma \in S} \sigma$. We call such a pair $(\mathcal{C}, S)$ a *colored category*. Moreover, a colored category $(\mathcal{C}, S)$ is called a *schemoid* if for a triple $\sigma, \tau, \mu \in S$ and for any morphisms f, g in μ, one has a bijection

$$(\pi^{\mu}_{\sigma\tau})^{-1}(f) \cong (\pi^{\mu}_{\sigma\tau})^{-1}(g), \tag{2.1}$$

where $\pi^{\mu}_{\sigma\tau} : \pi^{-1}_{\sigma\tau}(\mu) \to \mu$ denotes the restriction of the composition map