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Parity of sets of mutually orthogonal Latin squares ☆

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ABSTRACT

Every Latin square has three attributes that can be even or odd, but any two of these attributes determines the third. Hence the parity of a Latin square has an information content of 2 bits. We extend the definition of parity from Latin squares to sets of mutually orthogonal Latin squares (MOLS) and the corresponding orthogonal arrays (OA). Suppose the parity of an OA(k, n) has an information content of $\mathcal{B}(k, n)$ bits. We show that $\mathcal{B}(k, n) \leqslant \binom{k}{2} - 1$. For the case corresponding to projective planes we prove a tighter bound, namely $\mathcal{B}(n+1, n) \leqslant \binom{n}{2}$ when n is odd and $\mathcal{B}(n+1, n) \leqslant \binom{n}{2} - 1$ when n is even. Using the existence of MOLS with subMOLS, we prove that if $\mathcal{B}(k, n) = \binom{k}{2} - 1$ then $\mathcal{B}(k, N) = \binom{k}{2} - 1$ for all sufficiently large N.

Let the *ensemble* of an OA be the set of Latin squares derived by interpreting any three columns of the OA as a Latin square. We demonstrate many restrictions on the number of Latin squares of each parity that the ensemble of an OA(k, n) can contain. These restrictions depend on $n \pmod 4$ and give some insight as to why it is harder to build projective planes of order $n \equiv 2 \pmod 4$ than for $n \not\equiv 2 \pmod 4$. For example, we prove that when $n \equiv 2 \pmod 4$ it is impossible to build an OA$(n+1, n)$ for which all Latin squares in the ensemble are isotopic (equivalent to each other up to permutation of the rows, columns and symbols).

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1. Introduction

The aim of this paper is to develop a notion of parity for the orthogonal arrays that define sets of mutually orthogonal Latin squares (MOLS). The important notion of parity for permutations is widely known. A *Latin square* of order n is an $n \times n$ square with entries in an n-set Λ, called the alphabet, having the property that every element of Λ occurs exactly once in each row and each column of the square. Latin squares are 2-dimensional analogues of permutations and they too have a notion of parity. This parity plays a pivotal role in a famous conjecture of Alon and Tarsi (see e.g. [1,20] and the references therein) and has also proved crucial in a variety of other quite distinct investigations. In [7], parity was found to explain observed limitations on which Latin squares could be embedded together in topological surfaces. In [22] and [14], parity explains large components that arise in graphs made by local switchings in Latin squares or in 1-factorisations of the complete graph, respectively. Parity considerations can also assist in diagnosing symmetries of Latin squares [16].

It is clear that parity of single Latin squares is a useful concept. It is therefore natural to try to extend this concept to sets of MOLS. A set of MOLS is a set of Latin squares such that when any two of the squares are superimposed, every ordered pair of symbols occurs exactly once. For any list $M = [M_1, \dots, M_{k-2}]$ of MOLS on an alphabet Λ we define an $n^2 \times k$ matrix, denoted $\mathscr{A}(M)$, by taking one row $\big[r, c, M_1[r, c], \dots, M_{k-2}[r, c]\big]$ for each pair $(r, c) \in \Lambda^2$. (For the sake of definiteness, we insist that these rows are ordered lexicographically. Also, if M is given as a set rather than a list, then we impose lexicographic order on M in order to create $\mathscr{A}(M)$.) Now, $\mathscr{A}(M)$ is an *orthogonal array* $\mathrm{OA}(k, n)$ (of strength 2 with n levels and index 1) because it has the defining property that every pair of columns contains every ordered pair of elements of Λ exactly once. Conversely, if A is an $\mathrm{OA}(k, n)$ we define $\mathscr{M}(A)$ to be the set of $k-2$ MOLS formed by taking the entry in row r, column c of the i-th Latin square to be the entry in column $i+2$ of the row of A that begins $[r, c, \dots]$. In this sense, an $\mathrm{OA}(k, n)$ is equivalent to a set of $k-2$ MOLS of order n (see e.g. [6, III.3] for more details and background). Throughout this paper, we assume that k and n are integers and that $n \geqslant 2$ and $3 \leqslant k \leqslant n+1$.

Let $\mathcal{S}_\Lambda$ denote the permutations of Λ. Two orthogonal arrays on alphabet Λ are *isotopic* if one can be obtained from the other by some sequence of operations of the following type: choose a $\gamma \in \mathcal{S}_\Lambda$ and a column c and apply γ to every entry in column c. We say two orthogonal arrays are *conjugate* if one can be obtained from the other by permuting the columns. We say two Latin squares are isotopic (respectively, conjugate) if their orthogonal arrays are isotopic (respectively, conjugate). We say two orthogonal arrays are isomorphic if, up to possible reordering of the rows of the arrays, one is isotopic to a conjugate of the other.

A finite projective plane of order n (see e.g. [6] for the definition) can be used to define an $\mathrm{OA}(n+1, n)$ and vice versa. However, there are some subtleties to this relationship. Let $\mathcal{L}$ be a line of a finite projective plane of order n. We can make an $\mathrm{OA}(n+1, n)$, using $\mathcal{L}$ as