Journal of Combinatorial Theory, Series A 155 (2018) 244–266

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Cumulants for finite free convolution

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ARTICLE INFO

Article history:
Received 29 November 2016
Available online 17 November 2017

Keywords:
Cumulants
Finite free convolution
Free probability

ABSTRACT

In this paper we define cumulants for finite free convolution. We give a moment-cumulant formula and show that these cumulants satisfy desired properties: they are additive with respect to finite free convolution and they approach free cumulants as the dimension goes to infinity.

1. Introduction

Since the original paper of Voiculescu [18], where he discovered *asymptotic freeness*, Free Probability has given answers to many questions in random matrix theory in the *asymptotic regime*, see for example [1–4,6,7,14]. Recently, Marcus, Spielman, and Srivastava [9] found a connection between polynomial convolutions and addition of random matrices, which in the limit is related to free probability. The key idea is that instead of looking at distributions of eigenvalues of random matrices, one looks at the (expected) characteristic polynomial of a random matrix.

To be precise, for a matrix M, let $\chi_M(x) = \det(xI - M)$ be the characteristic polynomial of the matrix M. For d-dimensional hermitian matrices A and B with characteristic polynomials p and q, respectively, one defines the *finite free additive convolution* of p and q to be

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https://doi.org/10.1016/j.jcta.2017.11.012
0097-3165/

$$p(x) \boxplus_d q(x) = \mathbf{E}_Q[\chi_{A+QBQ^T}(x)],$$

where the expectation is taken over orthonormal matrices Q sampled according to the Haar measure. The convolution does not depend on the specific choice of A and B, but only on p and q.

The connection with free probability is that, as $d \to \infty$, this polynomial convolution approximates free additive convolution. The connection is quite remarkable since, as proved in [9], this convolution has appeared before and there are very explicit formulas to calculate the coefficients of the finite free convolution of two polynomials. Indeed, for $p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p$ and $q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^q$, the finite free convolution of p and q is given by

$$p(x) \boxplus_d q(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-i-j)!} a_i^p a_j^q.$$

A very successful approach to free probability is the combinatorial one developed by Speicher [15] based on free cumulants. He showed that the combinatorial structure of free cumulants is governed by non-crossing partitions. Hence, understanding the combinatorics on non-crossing partitions gave insight in many cases where analytic expressions could not be found.

In this paper we want to give a similar combinatorial treatment to finite free additive convolution. Our main contribution is describing cumulants for finite free additive convolution, by deriving moment-cumulant formulas. We will give alternative proofs of some of the results presented in [10]. Our approach has the advantage that it does not involve using random matrices and avoids the analytic machinery. Hence, this approach gives a different, combinatorial, understanding of finite free additive convolution. We also prove that these cumulants approximate free cumulants, when the degree of the polynomials tends to infinity.

Apart from this introduction, this paper is organized in five sections. In Section 2, we recall the preliminaries of non-crossing partitions, incidence algebras and free convolution. The main results are divided in Sections 3, 4 and 5. In Section 3 we introduce finite free cumulants and prove that they are additive. In Section 4 we give a moment-cumulant formula. In Section 5 we show that finite free cumulants approach free cumulants. Finally, in Section 6 we give some applications and examples.

2. Preliminaries

In this section we give the necessary preliminaries on incidence algebras, partitions, non-crossing partitions, and free cumulants. For a detailed account of the combinatorial theory of free probability we refer the reader to the monograph of Nica and Speicher [12].