Journal of Combinatorial Theory, Series A 155 (2018) 267–286

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

A sextuple equidistribution arising in Pattern Avoidance

Zhicong Lin [a,b], Dongsu Kim [c]

[a] School of Science, Jimei University, Xiamen 361021, PR China
[b] CAMP, National Institute for Mathematical Sciences, Daejeon, 34047, Republic of Korea
[c] Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, Daejeon, 34141, Republic of Korea

ARTICLE INFO

Article history:
Received 15 December 2016
Available online 20 November 2017

Keywords:
Double Eulerian distributions
Pattern Avoidance
Inversion sequences
Ascents
Descents
Schröder numbers

ABSTRACT

We construct an intriguing bijection between 021-avoiding inversion sequences and $(2413, 4213)$-avoiding permutations, which proves a sextuple equidistribution involving double Eulerian statistics. Two interesting applications of this result are also presented. Moreover, this result inspires us to characterize all permutation classes that avoid two patterns of length 4 whose descent polynomial equals that of separable permutations.

1. Introduction

Let $\mathbf{s} = \{s_i\}_{i\geq 1}$ be a sequence of positive integers. In order to interpret the Ehrhart polynomials of the $\mathbf{s}$-lecture hall polytopes, Savage and Schuster [20] introduced the associated $\mathbf{s}$-*inversion sequences of length* n as

$$\mathbf{I}_n^{(\mathbf{s})} := \{(e_1, e_2, \ldots, e_n) : 0 \leq e_i < s_i\}.$$

E-mail addresses: lin@nims.re.kr (Z. Lin), dongsu.kim@kaist.ac.kr (D. Kim).

https://doi.org/10.1016/j.jcta.2017.11.009

Since then, many remarkable equidistributions between $\mathbf{s}$-inversion sequences and permutations have been investigated [4,20,21,16]. In particular, Savage and Visontai [21] established connections with the descent polynomials of Coxeter groups which enable them to settle some real root conjectures. When $\mathbf{s} = (1, 2, 3, \dots)$, the $\mathbf{s}$-inversion sequences $\mathbf{I}_n = \mathbf{I}_n^{(\mathbf{s})}$ are usually called *inversion sequences*. Recently, Corteel–Martinez–Savage–Weselcouch [5] and Mansour–Shattuck [18] carried out the systematic studies of patterns in inversion sequences. A number of familiar combinatorial sequences, such as *large Schröder numbers* and *Euler numbers*, arise in their studies.

In this paper, we will prove a set-valued sextuple equidistribution arising from pattern avoidance in inversion sequences. Our result can be considered as an extension, since it involves four more statistics, of a restricted version of Foata's 1977 result [9], which asserts that descents and inverse descents on permutations have the same joint distribution as ascents and distinct entries on inversion sequences. We need some further notations and definitions before we can state our main result.

The inversion sequences $\mathbf{I}_n$ serve as various kinds of codings for $\mathfrak{S}_n$, the set of all permutations of $[n] := \{1, 2, \dots, n\}$. By a *coding* of $\mathfrak{S}_n$, we mean a bijection from $\mathfrak{S}_n$ to $\mathbf{I}_n$. For example, the map $\Theta(\pi) : \mathfrak{S}_n \to \mathbf{I}_n$ defined for $\pi = \pi_1\pi_2 \dots \pi_n \in \mathfrak{S}_n$ as

$$\Theta(\pi) = (e_1, e_2, \dots, e_n), \quad \text{where } e_i := \left|\{j : j < i \text{ and } \pi_j > \pi_i\}\right|,$$

is a natural coding of $\mathfrak{S}_n$. Clearly, the sum of the entries of $\Theta(\pi)$ equals the number of *inversions* of π, i.e., the number of pairs (i, j) such that $i < j$ and $\pi_i > \pi_j$. This is the reason why $\mathbf{I}_n$ is named inversion sequences here. For each $\pi \in \mathfrak{S}_n$ and each $e \in \mathbf{I}_n$, let

$$\mathrm{DES}(\pi) := \{i \in [n-1] : \pi_i > \pi_{i+1}\} \quad \text{and} \quad \mathrm{ASC}(e) := \{i \in [n-1] : e_i < e_{i+1}\}$$

be the ***des**cent set* of π and the ***asc**ent set* of e, respectively. Another important property of Θ is that $\mathrm{DES}(\pi) = \mathrm{ASC}(\Theta(\pi))$ for each $\pi \in \mathfrak{S}_n$. Thus,

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{DES}(\pi)} = \sum_{e \in \mathbf{I}_n} t^{\mathrm{ASC}(e)}, \tag{1.1}$$

where $t^S := \prod_{i \in S} t_i$ for any set S of positive integers.

Throughout this paper, we will use the convention that if upper case "ST" is a set-valued statistic, then lower case "st" is the corresponding numerical statistic. For example, $\mathrm{des}(\pi)$ is the cardinality of $\mathrm{DES}(\pi)$ for each $\pi \in \mathfrak{S}_n$. It is well known that $A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$ is the classical *n-th Eulerian polynomial* [10,22] and each statistic whose distribution gives $A_n(t)$ is called an *Eulerian statistic*. In view of (1.1), "asc" is an Eulerian statistic on inversion sequences. For each $e \in \mathbf{I}_n$, define $\mathrm{dist}(e)$ to be the *number of **dist**inct positive entries* of e. This statistic was first introduced by Dumont [7], who also showed that it is an Eulerian statistic on inversion sequences. Amazingly, Foata [9] later invented two different codings of permutations called *V-code* and *S-code* to prove the following extension of (1.1).