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A valency bound for distance-regular graphs

Zhi Qiao [a], Jack Koolen [b,*,1]

[a] College of Mathematics and Software Science, Sichuan Normal University, 610068, Sichuan, PR China
[b] School of Mathematical Sciences, University of Science and Technology of China, Wen-Tsun Wu Key Laboratory of the Chinese Academy of Sciences, 230026, Anhui, PR China

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ABSTRACT

The regular complete t-partite graphs $K_{t\times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$, and hence, for fixed t there are infinitely many of them. In this paper we will show that these graphs are exceptional graphs for the class of distance-regular graphs. For this we will show a valency bound for distance-regular graphs with a relatively large, in absolute value, smallest eigenvalue. Using this bound, we classify the non-bipartite distance-regular graphs with diameter at most three with smallest eigenvalue not larger than $-k/2$, where k is the valency of the graph. As an application we complete the classification of the 3-chromatic distance-regular graphs with diameter three, which was started by Blokhuis, Brouwer and Haemers.

* Corresponding author.
E-mail addresses: zhiqiao@sicnu.edu.cn (Z. Qiao), koolen@ustc.edu.cn (J. Koolen).
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1. Introduction

The regular complete t-partite graphs $K_{t\times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$, and hence, for fixed t there are infinitely many of them. (For notations and explanations of the graphs, see next section and [2] or [13].) In this paper we will show that these graphs are exceptional graphs for the class of distance-regular graphs. Namely, we will show the following valency bound for distance-regular graphs with a relatively large, in absolute value, smallest eigenvalue.

Theorem 1.1. *For any real number $\alpha \in (0,1)$ and any integer $D \geq 2$, there are finitely many coconnected non-bipartite distance-regular graphs with valency k at least two and diameter D, having smallest eigenvalue $\theta_{\min}$ not larger than $-\alpha k$.*

Remarks. (i) Note that the condition that the graphs are not bipartite is essential, as $-k$ is an eigenvalue of bipartite regular graphs with valency k and there are infinitely many bipartite distance-regular graphs with diameter 3, for example the point-block incidence graphs of projective planes of order q, where q is a prime power. For diameter 4 this is also true, for example, the Hadamard graphs.

(ii) Note that the same result is not true if you replace the absolute value of the smallest eigenvalue by the second largest eigenvalue. For example, the Johnson graph $J(n, D)$ $n \geq 2D \geq 4$, has valency $D(n-D)$ and second largest eigenvalue $(n-D-1)(D-1)-1$. So for fixed diameter $D \geq 3$, there are infinitely many Johnson graphs $J(n, D)$ with second largest eigenvalue larger than $k/2$.

(iii) The only distance-regular graphs that are not coconnected are the complete multipartite graphs.

We are aware of seven infinite families of non-bipartite distance-regular graphs with valency k, having smallest eigenvalue at most $-k/2$, including the complete tripartite graphs $K_{t,t,t}$. We think the following conjecture is true.

Conjecture 1.2. *When D is large enough, a non-bipartite distance-regular graph with valency k, diameter D, having smallest eigenvalue at most $-k/2$ is one of the following graphs*

1. *The odd polygons;*
2. *The folded $(2D+1)$-cubes;*
3. *The Odd graph O_k;*
4. *The Hamming graphs $H(D, 3)$;*
5. *The dual polar graphs of type $B_D(2)$;*
6. *The dual polar graphs of type ${}^2A_{2D-1}(2)$.*