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Symmetries of shamrocks, Part I ☆

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ABSTRACT

Hexagons with four-lobed regions called shamrocks removed from their center were introduced in their 2013 paper by Ciucu and Krattenthaler, where product formulas for the number of their lozenge tilings were provided. In analogy with the plane partitions which they generalize, we consider the problem of enumerating the lozenge tilings which are invariant under some symmetries of the underlying region. This leads to six symmetry classes besides the base case of requiring no symmetry. In this paper we provide product formulas for two of these symmetry classes (namely, the ones generalizing cyclically symmetric, and cyclically symmetric and transpose complementary plane partitions).

1. Introduction

In [10] we presented a generalization of MacMahon's classical counting of boxed plane partitions ([17], Sect. 495) — which is equivalent to enumerating lozenge tilings of hexagons on the triangular lattice — by providing product formulas for the number of lozenge tilings of hexagons from whose centers four-lobed structures called shamrocks were removed (such regions are called S-cored hexagons; see Fig. 1 for an example).

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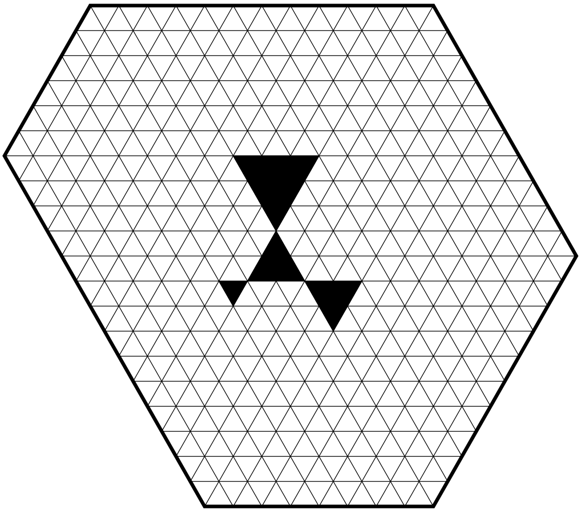

Fig. 1. The S-cored hexagon $SC_{6,8,4}(3,1,2,2)$ (see [10] for details of its definition).

Motivated by the case of plane partitions (see [1][19][16][20][12] and the surveys [2] and [13] for more recent developments) we consider in this paper the symmetry classes of tilings of S-cored hexagons. More precisely, for any subgroup G of the group of symmetries of an S-cored hexagon, we seek to find how many of its lozenge tilings are invariant under G.

This leads to six non-trivial symmetry classes, named after the corresponding symmetries of plane partitions (see [19]): (i) cyclically symmetric, (ii) transpose complementary, (iii) cyclically symmetric and transpose complementary, (iv) symmetric, (v) self-complementary and (vi) symmetric and self-complementary.

In this paper we provide product formulas for the first and third of these symmetry classes. The remaining ones require different methods and will be treated in subsequent work.

We point out that Kuo's graphical condensation method [14][15], which has been very fruitful in previous related work on enumerating tilings of lattice regions (see e.g. [5][10][6][4][7]), does not seem to provide the proofs here (the method *can* be applied to our regions, and it does lead to interesting recurrences, but they involve two different types of regions, and it does not seem possible to prove our formulas using them). Instead, we deduce our enumerations from two results we proved in [6] and two counterparts we present in this paper (we note that we need the full generality of the results in [6], and not just the special cases of [9] they extend). This constitutes an unforeseen and welcome application of the results of [6], which involve regions that did not seem particularly natural to consider, other than the fact that they extended certain regions of [9] in a general enough way so that conjectures concerning those regions could be proved by Kuo condensation.