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A graphical calculus for the Jack inner product on symmetric functions

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A B S T R A C T

Starting from a graded Frobenius superalgebra B, we consider a graphical calculus of B-decorated string diagrams. From this calculus we produce algebras consisting of closed planar diagrams and of closed annular diagrams. The action of annular diagrams on planar diagrams can be used to make clockwise (or counterclockwise) annular diagrams into an inner product space. Our main theorem identifies this space with the space of symmetric functions equipped with the Jack inner product at Jack parameter $\dim B_{\mathrm{even}} - \dim B_{\mathrm{odd}}$. In this way, we obtain a graphical realization of that inner product space.

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Contents

1. Introduction

Let B be a nonnegatively graded Frobenius superalgebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 (for example, the cohomology over $\mathbb{F}$ of a compact connected manifold). Inspired by constructions of Khovanov in [10] and Cautis and the first author in [4], the second and third author, in [19], associated to B a graded pivotal monoidal category $\mathcal{H}_B^*$. The objects of $\mathcal{H}_B^*$ are formal direct sums of compact oriented 0-manifolds, and the morphisms are linear combinations of immersed oriented planar 1-manifolds, decorated by elements of the Frobenius algebra B, and subject to certain local relations. Associated to $\mathcal{H}_B^*$ are at least two potentially interesting algebraic objects:

- the center $Z(\mathcal{H}_B^*)$, which is the endomorphism algebra of the monoidal unit of the category $\mathcal{H}_B^*$, is a graded supercommutative infinite-dimensional algebra, whose elements are linear combinations of immersed oriented closed 1-manifolds, decorated by elements of B, and subject to the local relations of the graphical calculus of $\mathcal{H}_B^*$;
- the trace, or zeroth Hochschild homology, $\mathrm{Tr}(\mathcal{H}_B^*)$ of $\mathcal{H}_B^*$ is a graded noncommutative infinite-dimensional algebra, whose elements are linear combinations of immersed closed annular 1-manifolds, decorated by elements of B, and subject to the local relations of the graphical calculus.

(We refer to Sections 3 and 4 for the precise definitions of the monoidal category $\mathcal{H}_B^*$ and the algebras $Z(\mathcal{H}_B^*)$ and $\mathrm{Tr}(\mathcal{H}_B^*)$.) The algebra of annular diagrams $\mathrm{Tr}(\mathcal{H}_B^*)$ acts linearly on the space of planar diagrams $Z(\mathcal{H}_B^*)$.

The importance of the trace in diagrammatic categorification is first emphasized in the work of Beliakova–Habiro–Lauda with both Webster [2] and with Guliyev [1]. In particular, the fact that the trace acts on the center of a pivotal monoidal category goes back at least to [1]. Nevertheless, at present not much is known about the algebra $\mathrm{Tr}(\mathcal{H}_B^*)$, except in the case $B = \mathbb{F}$, where it was computed in [6]. The authors there suggest that, in the general case, $\mathrm{Tr}(\mathcal{H}_B^*)$ should be understood in relation to the vertex algebra associated to the lattice $K_0(B)$, equipped with its Euler form. We do not take up the computation