Journal of Combinatorial Theory, Series A 154 (2018) 533–550

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Pattern avoidance and fiber bundle structures on Schubert varieties

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ARTICLE INFO

Article history:
Received 2 November 2016

Keywords:
Schubert variety
Flag variety
Grassmannian
Permutation
Pattern avoidance
Fiber bundle

ABSTRACT

We give a permutation pattern avoidance criteria for determining when the projection map from the flag variety to a Grassmannian induces a fiber bundle structure on a Schubert variety. In particular, we introduce the notion of a split pattern and show that a Schubert variety has such a fiber bundle structure if and only if the corresponding permutation avoids the split patterns $3|12$ and $23|1$. Continuing, we show that a Schubert variety is an iterated fiber bundle of Grassmannian Schubert varieties if and only if the corresponding permutation avoids (non-split) patterns 3412, 52341, and 635241. This extends a combined result of Lakshmibai–Sandhya, Ryan, and Wolper who prove that Schubert varieties whose permutation avoids the “smooth” patterns 3412 and 4231 are iterated fiber bundles of smooth Grassmannian Schubert varieties.

1. Introduction

A map $\pi : X \to Y$ between algebraic varieties is a fiber bundle with fiber F if for each point $y \in Y$, the fiber $\pi^{-1}(y)$ is isomorphic to F and there is a neighborhood U of y such that $\pi^{-1}(U) \simeq U \times F$. Fiber bundles are useful in the sense that the variety X is locally a product space. Hence many geometric and topological properties of X can be analyzed

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https://doi.org/10.1016/j.jcta.2017.09.006

by studying the factors F and Y. In this paper, we consider the projection map from the complete flag variety onto a Grassmannian. Specifically, let $\mathbb{K}$ be an algebraically closed field and let

$$F\ell(n) := \{V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{K}^n \mid \dim(V_i) = i\}$$

denote the complete flag variety on $\mathbb{K}^n$. For each $r \in \{1, \dots, n-1\}$, let

$$\mathrm{Gr}(r, n) := \{V \subset \mathbb{K}^n \mid \dim(V) = r\}$$

denote the Grassmannian of r-dimensional subspaces of $\mathbb{K}^n$ and consider the natural projection map

$$\pi_r : F\ell(n) \twoheadrightarrow \mathrm{Gr}(r, n) \tag{1}$$

given by $\pi_r(V_\bullet) = V_r$. It is easy to see that the projection π_r is a fiber bundle on $F\ell(n)$ with fibers isomorphic to $F\ell(r) \times F\ell(n-r)$. The goal of this paper is to give a pattern avoidance criteria for when the map π_r restricted to a Schubert variety of $F\ell(n)$ is also a fiber bundle.

Fix a basis $\{e_1, \dots, e_n\}$ of $\mathbb{K}^n$ and let $E_i := \mathrm{span}\langle e_1, \dots, e_i\rangle$. Each permutation $w = w(1) \cdots w(n)$ of the symmetric group $W := \mathfrak{S}_n$ defines a Schubert variety

$$X_w := \{V_\bullet \in F\ell(n) \mid \dim(E_i \cap V_j) \geq r_w[i, j]\}$$

where $r_w[i, j] := \#\{k \leq j \mid w(k) \leq i\}$.

Theorem 1.1. *Let $r < n$ and $w \in W$. The projection π_r restricted to X_w is a Zariski-locally trivial fiber bundle if and only if w avoids the split patterns $3|12$ and $23|1$ with respect to position r.*

If a permutation avoids a split pattern with respect to every position $r < n$, then that permutation avoids the pattern in the classical sense. For a precise definition of split pattern avoidance, see Definition 2.1. Pattern avoidance has been used to combinatorially describe many geometric properties of Schubert varieties. Most notably, Lakshmibai and Sandhya prove that a Schubert variety X_w is smooth if and only if w avoids the patterns 3412 and 4231 [9]. Pattern avoidance has also been used to determine when Schubert varieties are defined by inclusions, are Gorenstein, are factorial, have small resolutions, and are local complete intersections [3,6,7,4,16,14]. For a survey on Schubert varieties and their connections with pattern avoidance see [1].

1.1. Complete parabolic bundle structures

In this section we characterize when Schubert varieties have complete parabolic bundle structures. For a combinatorial analogue of this characterization see Section 2.1 on com-