Journal of Combinatorial Theory, Series A 154 (2018) 610–619

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Truncated theta series and a problem of Guo and Zeng

George E. Andrews [a], Mircea Merca [b]

[a] *Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA*
[b] *Academy of Romanian Scientists, Splaiul Independentei 54, Bucharest, 050094, Romania*

ARTICLE INFO

Article history:
Received 17 April 2017

Keywords:
Partitions
Euler's pentagonal number theorem
Theta series

ABSTRACT

We provide partition-theoretic interpretation of two truncated identities of Gauss solving a problem by Guo and Zeng. We also reveal that these results, together with our previous truncation of Euler's pentagonal number theorem, are essentially corollaries of the Rogers–Fine identity. Finally we examine further positivity questions related to the partition function.

1. Introduction

In [3], we proved the following identity for the partition function, $p(n)$:

Theorem 1. *For* $n > 0$, $k \geqslant 1$,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p\big(n-j(3j+1)/2\big)-p\big(n-j(3j+5)/2-1\big)\Big)=M_k(n), \qquad (1.1)$$

E-mail addresses: gea1@psu.edu (G.E. Andrews), mircea.merca@profinfo.edu.ro (M. Merca).

https://doi.org/10.1016/j.jcta.2017.10.003
0097-3165/

where $M_k(n)$ is the number of partitions of n in which k is the least integer that is not a part and there are more parts $> k$ than there are $< k$.

Yee [13] has given a combinatorial proof of Theorem 1. This theorem was directly deduced from the following:

Lemma 2. *For $k \geqslant 1$,*

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1}) = 1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad (1.2)$$

where

$$\begin{aligned}(A;q)_n &= \prod_{j=0}^{\infty}\frac{(1-Aq^j)}{(1-Aq^{j+n})} \\ &= \big((1-A)(1-Aq)\cdots(1-Aq^{n-1}) \quad \textit{if } n \textit{ is a positive integer}\big)\end{aligned}$$

and

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0, & \textit{if } B<0 \textit{ or } B>A \\ \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}, & \textit{otherwise.} \end{cases}$$

Apart from Euler's pentagonal number theorem ($k \to \infty$ in (1.2)):

$$(q;q)_\infty = \sum_{j=0}^{\infty}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1}), \quad (1.3)$$

there are two other central, classical theta identities (often attributed to Gauss and sometimes Jacobi) [2, p. 23, eqs. (2.2.12) and (2.2.13)]:

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = 1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2}, \quad (1.4)$$

and

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sum_{j=0}^{\infty}(-q)^{j(j+1)/2}. \quad (1.5)$$

We remark that the truncated theta series were recently studied in several papers by Guo and Zeng [7], Mao [10], Kolitsch [9], He, Ji and Zang [8], Chan, Ho and Mao [4], and Yee [13].

Guo and Zeng [7] proved analogues of the above Lemma 2 for (1.4) and (1.5). First they note that the reciprocal of the infinite product in (1.4) is the generating function