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Turán number of generalized triangles

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ABSTRACT

The family Σ_r consists of all r-graphs with three edges D_1, D_2, D_3 such that $|D_1 \cap D_2| = r - 1$ and $D_1 \triangle D_2 \subseteq D_3$. A *generalized triangle*, $\mathcal{T}_r \in \Sigma_r$ is an r-graph on $\{1, 2, \ldots, 2r-1\}$ with three edges D_1, D_2, D_3, such that $D_1 = \{1, 2, \ldots, r-1, r\}$, $D_2 = \{1, 2, \ldots, r-1, r+1\}$ and $D_3 = \{r, r+1, \ldots, 2r-1\}$. Frankl and Füredi conjectured that for all $r \geq 4$, $\mathrm{ex}(n, \Sigma_r) = \mathrm{ex}(n, \mathcal{T}_r)$ for all sufficiently large n and they also proved it for $r = 3$. Later, Pikhurko showed that the conjecture holds for $r = 4$. In this paper we determine $\mathrm{ex}(n, \mathcal{T}_5)$ and $\mathrm{ex}(n, \mathcal{T}_6)$ for sufficiently large n, proving the conjecture for $r = 5, 6$.

1. Introduction

In this paper we consider r-uniform hypergraphs, which we call r-graphs for brevity. We denote the vertex set of an r-graph $\mathcal{G}$ by $V(\mathcal{G})$ and the number of its vertices by $\mathrm{v}(\mathcal{G})$. Let $\mathfrak{F}$ be a family of r-graphs. An r-graph $\mathcal{G}$ is $\mathfrak{F}$-free if it does not contain any member of $\mathfrak{F}$ as a subgraph. The Turán function $\mathrm{ex}(n, \mathfrak{F})$ is the maximum size of an $\mathfrak{F}$-free r-graph of order n:

$$\mathrm{ex}(n, \mathfrak{F}) = \max\left\{|\mathcal{G}| : \mathrm{v}(\mathcal{G}) = n \text{ and } \mathcal{G} \text{ is } \mathfrak{F}\text{-free}\right\}.$$

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When $\mathfrak{F}$ contains just one element, say $\mathfrak{F} := \{\mathcal{F}\}$, we write $\text{ex}(n, \mathcal{F}) := \text{ex}(n, \mathfrak{F})$. Let $\mathfrak{T}_r$ be the family of all r-graphs with three edges such that one edge contains the symmetric difference of the other two, and let Σ_r be the family of all r-graphs with three edges D_1, D_2, D_3 such that $|D_1 \cap D_2| = r-1$ and $D_1 \triangle D_2 \subseteq D_3$. A *generalized triangle*, $\mathcal{T}_r \in \Sigma_r$ is an r-graph on $[2r-1]$ with three edges D_1, D_2, D_3, such that $D_1 = \{1, 2, \ldots, r-1, r\}, D_2 = \{1, 2, \ldots, r-1, r+1\}$ and $D_3 = \{r, r+1, \ldots, 2r-1\}$.

Note that $\mathcal{T}_r \in \Sigma_r$ and $\Sigma_r \subseteq \mathfrak{T}_r$. Note also that for $r = 2, 3$, $\Sigma_r = \mathfrak{T}_r$. As a generalization of Turán's theorem, in [8] Katona suggested to determine $\text{ex}(n, \mathfrak{T}_3)$. This question was answered by Bollobás in [2]. He showed that for any $n \geq 3$ the complete balanced 3-partite 3-graph (that is, the sizes of any two parts differ at most by one) is the unique extremal graph. Hence,

$$\text{ex}(n, \mathfrak{T}_3) = \left\lfloor \frac{n}{3} \right\rfloor \times \left\lfloor \frac{n+1}{3} \right\rfloor \times \left\lfloor \frac{n+2}{3} \right\rfloor.$$

Bollobás conjectured that the same result holds for all $r \geq 4$. In [16] Sidorenko proved the conjecture for $r = 4$, in fact he showed that $\text{ex}(n, \mathfrak{T}_4) = \text{ex}(n, \Sigma_4)$ and determined the latter. However, Shearer [15] showed that Bollobás conjecture fails for $r \geq 10$.

But what can be said about the relation between $\text{ex}(n, \mathcal{T}_r)$ and $\text{ex}(n, \Sigma_r)$? In [3], Erdős and Simonovits proved that for any fixed r

$$\text{ex}(n, \mathcal{T}_r) - \text{ex}(n, \Sigma_r) = o(n^r).$$

Later, in [5] Frankl and Füredi conjectured that these numbers are the same for sufficiently large n.

Conjecture 1.1 *(P. Frankl, Z. Füredi, [5]). For every $r \geq 4$, there exists $n_0 := n_0(r)$ such that for all $n \geq n_0$*

$$\text{ex}(n, \mathcal{T}_r) = \text{ex}(n, \Sigma_r).$$

In their previous work [4], Frankl and Füredi showed that Conjecture 1.1 holds for $r = 3$ with very large n_0. Keevash and Mubayi [10] presented a different proof of this result; they showed that one can take $n_0 = 33$. Recently, the conjecture for $r = 4$ was proved by Pikhurko [13]. We show that the conjecture holds for $r = 5, 6$.

Theorem 1.1. *There exists n_0 such that for all $n \geq n_0$, $\text{ex}(n, \mathcal{T}_r) = \text{ex}(n, \Sigma_r)$ for $r = 5, 6$. Moreover, extremal graphs are blowups of the unique $(11, 5, 4)$ and $(12, 6, 5)$ Steiner systems for $r = 5$ and $r = 6$, respectively.*

For general $r \geq 7$ the problem of determining $\text{ex}(n, \Sigma_r)$, even asymptotically, remains open. The following bounds were proved in [5].