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Three-coloring triangle-free graphs on surfaces II. 4-critical graphs in a disk ☆

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Let G be a plane graph of girth at least five. We show that if there exists a 3-coloring ϕ of a cycle C of G that does not extend to a 3-coloring of G, then G has a subgraph H on $O(|C|)$ vertices that also has no 3-coloring extending ϕ. This is asymptotically best possible and improves a previous bound of Thomassen. In the next paper of the series we will use this result and the attendant theory to prove a generalization to graphs on surfaces with several precolored cycles.

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1. Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Historically the first result in this direction is the following classical theorem of Grötzsch [8].

Theorem 1.1. *Every triangle-free planar graph is* 3*-colorable.*

Thomassen [13,14,16] found three reasonably simple proofs of this statement. Recently, two of us, in joint work with Kawarabayashi [3] were able to design a linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 1.1. Another significantly different proof was given by Kostochka and Yancey [10].

The statement of Theorem 1.1 cannot be directly extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely many 4-critical triangle-free graphs that can be drawn in Σ. (A graph is 4-*critical* if it is not 3-colorable, but every proper subgraph is.) For instance, the graphs obtained from an odd cycle of length five or more by applying Mycielski's construction [1, Section 8.5] have that property. Thus an algorithm for testing 3-colorability of triangle-free graphs on a fixed surface will have to involve more than just testing the presence of finitely many obstructions.

The situation is different for graphs of girth at least five by another deep theorem of Thomassen [15], the following.

Theorem 1.2. *For every surface Σ there are only finitely many* 4*-critical graphs of girth at least five that can be drawn in Σ.*

Thus the 3-colorability problem on a fixed surface has a polynomial-time algorithm for graphs of girth at least five, but the presence of cycles of length four complicates matters. Let us remark that there are no 4-critical graphs of girth at least five on the projective plane and the torus [13] and on the Klein bottle [12].

The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized is the projective plane. Building on earlier work of Youngs [18], Gimbel and Thomassen [7] obtained the following elegant characterization. A graph drawn in a surface is a *quadrangulation* if every face is bounded by a cycle of length four.

Theorem 1.3. *A triangle-free graph drawn in the projective plane is* 3*-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

For other surfaces there does not seem to be a similarly nice characterization. Gimbel and Thomassen [7, Problem 3] asked whether there is a polynomial-time algorithm to test