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Excluding a large theta graph

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ABSTRACT

A *theta graph*, denoted $\theta_{a,b,c}$, is a graph of order $a+b+c-1$ consisting of a pair of vertices and three internally-disjoint paths between them of lengths a, b, and c. In this paper we study graphs that do not contain a large $\theta_{a,b,c}$ minor. More specifically, we describe the structure of $\theta_{1,2,t}$-, $\theta_{2,2,t}$-, $\theta_{1,t,t}$-, $\theta_{2,t,t}$-, and $\theta_{t,t,t}$-free graphs where t is large. The main result is a characterization of $\theta_{t,t,t}$-free graphs for large t. The 3-connected $\theta_{t,t,t}$-free graphs are formed by 3-summing graphs without a long path to certain planar graphs. The 2-connected $\theta_{t,t,t}$-free graphs are then built up in a similar fashion by 2- and 3-sums. This result implies a well-known theorem of Robertson and Chakravarti on graphs that do not have a bond containing three specified edges.

1. Introduction

All graphs are loopless but may have parallel edges. Undefined terminology can be found in [1]. In particular, a set of paths is called *independent* if no interior vertex of one path is contained in another path.

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In this paper, we describe the structure of graphs that do not contain certain large theta graphs as a minor. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A *theta graph*, denoted $\theta_{a,b,c}$, is a graph of order $a+b+c-1$ consisting of a pair of vertices and three independent paths between them of lengths a, b, and c. Theta graphs have maximum degree 3 so containing a theta graph as a minor is equivalent to containing a theta graph as a topological minor. Throughout we will say *G contains $\theta_{a,b,c}$* to mean G contains $\theta_{a,b,c}$ as a minor (or topological minor). Additionally we use the phrase *G contains a $\theta_{a,b,c}$ graph at u and v* to mean G contains as a subgraph a subdivision of $\theta_{a,b,c}$ in which u and v are the two vertices of degree 3. A graph is *$\theta_{a,b,c}$-free* if it does not contain $\theta_{a,b,c}$.

The main goal of this paper is to describe all $\theta_{t,t,t}$-free graphs for large integers t. In other words, we want to characterize all graphs that do not contain three long independent paths between any pair of vertices. This problem is in fact an instance of a very general problem stated below.

Problem (P): *for a given infinite set $\mathcal{H} = \{H_1, H_2, \ldots,\}$ of graphs where each H_i is a minor of H_{i+1}, characterize all minor-closed classes $\mathcal{G}$ of graphs for which $\mathcal{G} \not\supseteq \mathcal{H}$.*

Note $\mathcal{G} \not\supseteq \mathcal{H}$ holds if and only if all graphs in $\mathcal{G}$ are H_i-free for some i. Since this i can be (and can also be assumed to be) arbitrarily big, problem (P) is often loosely stated as: characterize graphs that do not contain a large H_i minor. Therefore when we say we want to characterize graphs that do not have a *large* $\theta_{t,t,t}$ minor, we are talking about solving (P) for $\mathcal{H} = \{\theta_{t,t,t} : t \geq 2\}$. Likewise, to characterize $\theta_{1,2,t}$-free graphs for large t means we are solving (P) for $\mathcal{H} = \{\theta_{1,2,t} : t \geq 2\}$.

There are several choices of $\mathcal{H}$ for which (P) has been solved. Along this line, the best known results are the two obtained by Robertson and Seymour which solve (P) for the class of all complete graphs [10] and for the class of all planar grids [8]. The same authors also solved (P) for the classes of all trees, all stars, and all paths [7,11]. Other classes for which (P) is solved include the class of all wheels [3] and the class of all double-paths [2].

We prove that $\theta_{t,t,t}$-free graphs have the following structure: begin with a planar graph that contains no long paths outside of a special facial cycle and attach graphs that do not have long paths to the planar graph along edges, facial triangles, and certain facial 4-cycles. This result is stated formally in the next section. Additionally, we obtain corresponding results for all $\theta_{1,2,t}$-, $\theta_{2,2,t}$-, $\theta_{1,t,t}$-, and $\theta_{2,t,t}$-free graphs.

Our result for $\theta_{t,t,t}$-free graphs implies a result of Robertson and Chakravarti [6] concerning when three specified edges of a graph are contained in a *bond* (a minimal nonempty edge-cut of the graph). Suppose we subdivide the three specified edges sufficiently many times. Then it is easy to see that the three specified edges are contained in a bond in the original graph if and only if the subdivided graph contains $\theta_{t,t,t}$. This connection easily leads to the result of Robertson and Chakravarti, as we will see in the next section, and it also illustrates how much our result strengthens their result.

Another important goal of this paper is to develop tools for dealing with various cases of problem (P). We will prove several key lemmas that could be used in similar

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