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Linearly many rainbow trees in properly edge-coloured complete graphs

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ABSTRACT

A subgraph of an edge-coloured complete graph is called rainbow if all its edges have different colours. The study of rainbow decompositions has a long history, going back to the work of Euler on Latin squares. In this paper we discuss three problems about decomposing complete graphs into rainbow trees: the Brualdi–Hollingsworth Conjecture, Constantine's Conjecture, and the Kaneko–Kano–Suzuki Conjecture. We show that in every proper edge-colouring of K_n there are $10^{-6}n$ edge-disjoint spanning isomorphic rainbow trees. This simultaneously improves the best known bounds on all these conjectures. Using our method we also show that every properly $(n-1)$-edge-coloured K_n has $n/9-6$ edge-disjoint rainbow trees, giving further improvement on the Brualdi–Hollingsworth Conjecture.

1. Introduction

In this paper we consider the following question: Can the edges of every properly edge-coloured complete graph be decomposed into edge-disjoint rainbow spanning trees.

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Here a properly edge-coloured complete graph K_n means an assignment of colours to the edges of K_n so that no two edges at a vertex receive the same colour. A rainbow spanning tree in K_n is a tree containing every vertex of K_n, all of whose edges have different colours.

The study of rainbow decompositions dates back to the 18th century when Euler studied the question "for which n does there exist a properly n-edge-coloured $K_{n,n}$ which can be decomposed into n edge-disjoint rainbow perfect matchings.[2]" Euler constructed such proper n-edge-colourings of $K_{n,n}$ whenever $n \not\equiv 2 \pmod 4$, and conjectured that these are the only values of n for which they can exist. The $n = 6$ case of this conjecture is Euler's famous "36 officers problem", which was eventually proved by Tarry in 1901. For larger n, Euler's Conjecture was disproved in 1959 by Parker, Bose, and Shrikhande. Together these results give a complete description of the values of n for which there exists a properly n-edge-coloured $K_{n,n}$ which can be decomposed into n edge-disjoint rainbow perfect matchings.

Decompositions of properly $(2n-1)$-edge-coloured K_{2n} into edge-disjoint rainbow perfect matchings have also been studied. They were introduced by Room in 1955,[3] who raised the question of which n they exist for. Wallis showed that such decompositions of K_{2n} exist if, and only if, $n \neq 2$ or 4. Rainbow perfect matching decompositions of both $K_{n,n}$ and K_{2n} have found applications in scheduling tournaments and constructing experimental designs (see e.g. [10]).

Euler and Room wanted to determine the values of n for which there exist colourings of $K_{n,n}$ or K_n with rainbow matching decompositions. However given an *arbitrary* proper edge-colouring of $K_{n,n}$ or K_n it is not the case that it must have a decomposition into rainbow perfect matchings. A natural way of getting around this is to consider decompositions into rainbow graphs other than perfect matchings. In the past decompositions into rainbow subgraphs such as cycles and triangle factors have been considered [8].

An additional reason to study rainbow subgraphs arises in Ramsey theory, more precisely in the canonical version of Ramsey's theorem, proved by Erdős and Rado [11] in 1950. Here the goal is to show that edge-colourings of K_n, in which each colour appears only few times contain rainbow copies of certain graphs (see, e.g., introduction of [19], for more details).

In this paper we consider decompositions into rainbow trees. In contrast to the perfect matching case, it is believed that every properly edge coloured K_n can be decomposed into edge-disjoint rainbow trees. This was conjectured by three different sets of authors.

[2] Euler studied the values of n for which a pair of $n \times n$ orthogonal Latin squares exists. Using a standard argument, it is easy to show that $n \times n$ orthogonal Latin squares are equivalent objects to rainbow perfect matching decompositions of $K_{n,n}$.

[3] Room actually introduced objects which are now called "Room squares". It is easy to show that Room squares are equivalent objects to decompositions of $(2n-1)$-edge-coloured K_{2n} into edge-disjoint rainbow perfect matchings.