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The treewidth of line graphs

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ABSTRACT

The treewidth of a graph is an important invariant in structural and algorithmic graph theory. This paper studies the treewidth of *line graphs*. We show that determining the treewidth of the line graph of a graph G is equivalent to determining the minimum vertex congestion of an embedding of G into a tree. Using this result, we prove sharp lower bounds in terms of both the minimum degree and average degree of G. These results are precise enough to exactly determine the treewidth of the line graph of a complete graph and other interesting examples. We also improve the best known upper bound on the treewidth of a line graph. Analogous results are proved for pathwidth.

1. Introduction

Treewidth is a graph parameter that measures how "tree-like" a graph is. It is of fundamental importance in structural graph theory (especially in the graph minor theory of Robertson and Seymour [23]) and in algorithmic graph theory, since many NP-complete problems are solvable in polynomial time on graphs of bounded treewidth [4]. Let $\mathsf{tw}(G)$ denote the treewidth of a graph G (defined below). This paper studies the treewidth of

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line graphs. For a graph G, the line graph $L(G)$ is the graph with vertex set $E(G)$ where two vertices are adjacent if and only if their corresponding edges are incident. (We shall refer to vertices in the line graph as edges—vertices shall refer to the vertices of G itself unless explicitly noted.)

As a concrete example, the treewidth of $L(K_n)$ is important in recent work by Grohe and Marx [12] and Marx [21]. Specifically, Marx [21] showed that if $\mathsf{tw}(G) \geq k$ then the lexicographic product of G with K_p contains the lexicographic product of $L(K_k)$ with K_q as a minor (for choices of p and q depending on $|V(G)|$ and k). Motivated by this result, the authors determined the treewidth of $L(K_n)$ exactly [14]. The techniques used were extended to determine the treewidth of the line graph of a complete multipartite graph up to lower order terms, with an exact result when the complete multipartite graph is regular [13]. These results also extend to pathwidth (since the tree decompositions constructed have paths as the underlying trees).

Lower bounds The following are two elementary lower bounds on $\mathsf{tw}(L(G))$. First, if $\Delta(G)$ is the maximum degree of G, then $\mathsf{tw}(L(G)) \geq \Delta(G) - 1$ since the edges incident to a vertex in G form a clique in $L(G)$. Second, given a minimum width tree decomposition of $L(G)$, replace each edge with both of its endpoints to obtain a tree decomposition of G. It follows that

$$\mathsf{tw}(L(G)) \geq \tfrac{1}{2}(\mathsf{tw}(G) + 1) - 1. \tag{1}$$

We prove the following lower bound on $\mathsf{tw}(L(G))$ in terms of $\mathsf{d}(G)^2$, where $\mathsf{d}(G)$ is the average degree of G.

Theorem 1.1. *For every graph G with average degree $\mathsf{d}(G)$,*

$$\mathsf{pw}(L(G)) \geq \mathsf{tw}(L(G)) > \tfrac{1}{8}\mathsf{d}(G)^2 + \tfrac{3}{4}\mathsf{d}(G) - 2.$$

The bound in Theorem 1.1 is within '+1' of optimal since we show that for all k and n there is an n-vertex graph G with $\mathsf{d}(G) \approx 2k$ and $\mathsf{tw}(L(G)) \leq \mathsf{pw}(L(G)) = \frac{1}{8}(2k)^2 + \frac{3}{4}(2k) - 1$. All these results are proven in Section 3.

We also prove a sharp lower bound in terms of $\delta(G)^2$, where $\delta(G)$ is the minimum degree of G. (The constants in Theorem 1.1 and 1.2 are such that, depending on the graph, either result could be stronger.)

Theorem 1.2. *For every graph G with minimum degree $\delta(G)$,*

$$\mathsf{pw}(L(G)) \geq \mathsf{tw}(L(G)) \geq \begin{cases} \frac{1}{4}\delta(G)^2 + \delta(G) - 1 & \text{when } \delta(G) \text{ is even} \\ \frac{1}{4}\delta(G)^2 + \delta(G) - \frac{5}{4} & \text{when } \delta(G) \text{ is odd.} \end{cases}$$

The bound in Theorem 1.2 is sharp since for all n and k we describe a graph G with n vertices and minimum degree k such that $\mathsf{pw}(L(G))$ equals the bound in Theorem 1.2