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Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes

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A B S T R A C T

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. We prove that for all $\nu > 0$, every triangle-free graph with sufficiently large chromatic number contains holes of ν consecutive lengths.

1. Introduction

All graphs in this paper are finite and without loops or parallel edges. A *hole* in a graph is an induced subgraph which is a cycle of length at least four, and a hole is *odd* if its length is odd. A *triangle* in G is a three-vertex complete subgraph, and a graph is *triangle-free* if it has no triangle. In this paper we are concerned with the chromatic number of triangle-free graphs that have no holes of certain specified lengths.

What can we say about the hole lengths in triangle-free graphs with large chromatic number? There are three well-known conjectures of Gyárfás [6], the third implying the first two, as follows:

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1.1 Conjecture. *For all k, ℓ, there exists n such that if G has no clique of cardinality k and has chromatic number at least n, then*

- *G has an odd hole;*
- *G has a hole of length at least ℓ; and*
- *G has an odd hole of length at least ℓ.*

The first conjecture was proved in [7], and the second in [4]. There are a few other results about the lengths of holes in a graph G with (sufficiently) large chromatic number:

- G contains a large clique or an even hole [1];
- G contains a large clique or a hole of length 5 or a long hole [3];
- G contains a triangle or an odd hole of length at least seven [3]; and
- G contains a triangle or a hole of length a multiple of three [2].

Since this paper was submitted, there has been some further progress. In joint work with Maria Chudnovsky and Sophie Spirkl [5], we proved the third conjecture of Gyárfás. Finally, in recent work [8], we proved the following result, which gives a common generalization of all the results mentioned above.

1.2. *For all k, s, t, there exists n such that if G has no clique of cardinality k and has chromatic number at least n, then G has a hole of length s modulo t.*

In this paper we consider the case $k = 3$. In this case, we show that a far stronger result holds. The main result of this paper is:

1.3. *For all integers $\nu > 0$ there exists n such that if G is triangle-free with chromatic number at least n, then for some t, G has a hole of length $t + i$ for $1 \leq i \leq \nu$.*

This contains as special cases the $k = 3$ cases of all the results mentioned above. We conjecture that the corresponding result is true for graphs with bounded clique number rather than just triangle-free graphs, but so far we have made no progress in proving this.

Let us mention in passing a much more general question, which seems to be interesting even though we cannot answer it. Let us say a set F of integers is *k-constricting* if there exists n such that every graph with chromatic number at least n contains either a clique with k vertices or a hole with length in F. Say that F is *constricting* if it is k-constricting for every k. Which sets are constricting? Certainly every constricting set is infinite, because there are graphs with arbitrarily large chromatic number and arbitrarily large girth. On the other hand, a consequence of our main result is the following.

1.4. *Let F be an infinite set of positive integers with bounded gaps. Then F is 3-constricting.*