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# A stability theorem for maximal $K_{r+1}$-free graphs

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ABSTRACT

For $r \geq 2$, we show that every maximal $K_{r+1}$-free graph $G$ on $n$ vertices with $(1 - \frac{1}{r})\frac{n^2}{2} - o(n^{\frac{r+1}{r}})$ edges contains a complete $r$-partite subgraph on $(1 - o(1))n$ vertices. We also show that this is best possible. This result answers a question of Tyomkyn and Uzzell.



## 1. Introduction

For a positive integer $r \geq 2$, a graph $G$ is said to be $(r+1)$-*saturated* (or *maximal* $K_{r+1}$-*free*) if it contains no copy of $K_{r+1}$, but the addition of any edge from the complement $\overline{G}$ creates at least one copy of $K_{r+1}$. Let $T_r(n)$ denote the $r$-*partite Turán graph* that is, the $n$-vertex, complete $r$-partite graph for which each of the $r$ classes is of order $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. We write $t_r(n) = e(T_r(n))$, and note that $t_r(n) = (1 - \frac{1}{r})\frac{n^2}{2} + O_r(1)$. Whenever we speak of an $r$-partite subgraph, we require that it is induced.

The classical theorem of Turán [12] tells us that, for an integer $r \geq 2$, the maximum number of edges in a graph not containing a $K_{r+1}$ is $t_r(n)$, and that $T_r(n)$ is the unique $K_{r+1}$-free graph attaining this maximum. Erdős and Simonovits [6,5,11] discovered that

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this extremal problem exhibits a certain 'stability' phenomenon: $K_{r+1}$-free graphs for which $e(G)$ is close to $t_r(n)$ must resemble the Turán graph in an appropriate sense. In particular, they proved that every $n$-vertex, $K_{r+1}$-free graph with at least $t_r(n) - o(n^2)$ edges can be transformed into $T_r(n)$ by making at most $o(n^2)$ edge deletions and additions.

Beyond the seminal work of Erdős and Simonovits, we are lead to consider finer aspects of this phenomenon. More generally, it is natural to ask how the structure of a $K_{r+1}$-free graph $G$ comes to resemble the Turán graph as the number of edges $e(G)$ approaches the Turán number $t_r(n)$. For instance, Nikiforov and Rousseau [10], in the context of a Ramsey-theoretic problem, showed that for $r \geq 2$ and $\varepsilon$ sufficiently small (depending on $r$) the following holds: if $G$ is an $n$-vertex $K_{r+1}$-free graph with $e(G) \geq \left(1 - \frac{1}{r} - \varepsilon\right) n^2/2$, then $G$ contains an induced $r$-partite subgraph $H$ with $|H| \geq (1 - 2\varepsilon^{1/3})n$ and $\delta(H) \geq \left(1 - \frac{1}{r} - 4\varepsilon^{1/3}\right) n$. In other words, $G$ must contain a large $r$-partite subgraph with minimum degree almost as large as $\delta(T_r(n))$. The interested reader should consult the survey of Nikiforov [9] for a few other stability results in a similar vein.

Another result concerning the finer structure of stability is due to Brouwer [4], who showed that if $n \geq 2r + 1$ and $G$ is a $K_{r+1}$-free graph with $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then $G$ must be $r$-partite. This result has further been rediscovered by several authors [1,7,8], and Tyomkyn and Uzzell [13] recently gave a new proof. In this paper, we are interested in the structure of *maximal* $K_{r+1}$-free graphs near the Turán threshold. In this context, Brouwer's result says that if the number of edges of an $(r + 1)$-saturated graph $G$ is roughly within $n/r$ of the Turán number $t_r(n)$, then $G$ *is* complete $r$-partite. A natural question then arises, which informally is: when can one guarantee 'almost-spanning' complete $r$-partite subgraphs in $(r + 1)$-saturated graphs?

Continuing this line of investigation, Tyomkyn and Uzzell [13] proved, among other results, that every 4-saturated graph on $n$ vertices and with $t_3(n) - cn$ edges contains a complete 3-partite graph on $(1 - o(1))n$ vertices (they also implicitly dealt with the 3-saturated case). They went on to ask if one can similarly find almost-spanning, complete $r$-partite subgraphs in $(r + 1)$-saturated graphs with many edges, for $r \geq 4$. The main result of this paper is to resolve the question of Tyomkyn and Uzzell, in a stronger form. Not only do we show that this phenomenon persists for $(r + 1)$-saturated graphs for all $r \geq 2$, but we also determine the edge threshold for which the result fails to hold. In particular, we show the following.

**Theorem 1.1.** *Let $r \geq 2$ be an integer. Every $(r+1)$-saturated graph $G$ on $n$ vertices with $t_r(n) - o(n^{\frac{r+1}{r}})$ edges contains a complete $r$-partite subgraph on $(1 - o(1))n$ vertices.*

We also show that this theorem is tight in the sense that for every $\delta > 0$ there exist graphs $G$ with $t_r(n) - \delta n^{\frac{r+1}{r}}$ edges for which the conclusion of Theorem 1.1 fails.

We actually deduce Theorem 1.1 from a stronger, quantitative result, which we now make precise. For a graph $G$ and an integer $r \geq 2$, define the graph parameter

$$g_r(G) = \min\{|T| : T \subseteq V(G), G - T \text{ is complete } r\text{-partite}\}.$$

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