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Decomposing a graph into forests and a matching

Daqing Yang, [a,b,1]

[a] *Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, China*
[b] *Center for Discrete Mathematics, Fuzhou University, Fuzhou, Fujian, China*

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ABSTRACT

The fractional arboricity of a graph G, denoted by $\gamma_f(G)$, is defined as $\gamma_f(G) = \max_{H \subseteq G, v(H)>1} \frac{e(H)}{v(H)-1}$. The famous Nash-Williams' Theorem states that a graph G can be partitioned into at most k forests if and only if $\gamma_f(G) \leq k$. A graph is d-bounded if it has maximum degree at most d. The Nine Dragon Tree (NDT) Conjecture [posed by Montassier, Ossona de Mendez, Raspaud, and Zhu, at [11]] asserts that if $\gamma_f(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests with one being d-bounded. In this paper, it is proven that the Nine Dragon Tree Conjecture is true for all the cases in which $d = 1$.

1. Introduction

Throughout this paper, graphs are finite. A graph may have parallel edges but no loops. Given a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Let $v(G) := |V(G)|$ and $e(G) := |E(G)|$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X; $G - X$ is the graph obtained from G by deleting vertices

E-mail addresses: dyang@zjnu.edu.cn, daqing85@yahoo.com.
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in X. If S is a set of edges, the edge-induced subgraph $G[S]$ is the subgraph of G whose edge set is S and whose vertex set consists of all ends of edges of S. Let $N_G(v)$ denote the neighbors of v, $E_G(v)$ denote the edges that are incident to v, and let $d_G(v)$ be the degree of v, i.e., $d_G(v) = |E_G(v)|$.

Given a graph G, a *decomposition* of a graph G consists of edge-disjoint subgraphs with union G. The *arboricity* of G, denoted $\gamma(G)$, is the minimum number of forests needed to decompose it. The *fractional arboricity* of G is defined as

$$\gamma_f(G) = \max_{H \subseteq G, v(H) > 1} \frac{e(H)}{v(H) - 1}.$$

This notion was introduced by Payan [14]; see also [2,6]. The famous Nash-Williams' Theorem states a necessary and sufficient condition for $\gamma(G) \le k$:

Theorem 1.1 *(Nash-Williams 1964 [13]). A graph $G = (V, E)$ can be decomposed into at most k forests if and only if $\gamma_f(G) \le k$.*

This is a sparseness condition. Say a graph is *d-bounded* if it has maximum degree at most d. Montassier et al. [11] posed *the Nine Dragon Tree (NDT) Conjecture* (honoring a famous tree in Kaohsiung, Taiwan that is far from acyclic):

Conjecture 1.2 *(The NDT (Nine Dragon Tree) Conjecture [11]). Suppose G is a graph and k, d are non-negative integers. If $\gamma_f(G) \le k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests with one being d-bounded.*

Let a (k, d)-decomposition of a graph G be a decomposition of G into $k + 1$ forests with one having maximum degree at most d. Graphs having such a decomposition are (k, d)-decomposable. Regarding the progress of Nine Dragon Tree (NDT) Conjecture, Montassier et al. [11] proved the cases of $(k, d) = (1, 1)$ and $(k, d) = (1, 2)$, and they showed that no larger value of $\gamma_f(G)$ is sufficient. Kim et al. [10] proved the cases of $(k, d) = (k, k + 1)$ and $(k, d) = (1, d)$ for $d \le 6$. Chen et al. [3] proved the cases (k, d) for all d when $k \le 2$, except for $(k, d) = (2, 1)$. By using an extension of the matroid intersection theorem [4] due to Aharoni and Berger [1], Kaiser et al. in [9] proved that, if $\gamma_f(G) \le k + \frac{1}{3k+2}$, then G is $(k, 1)$-decomposable.

In this paper, we shall prove that, if $\gamma_f(G) \le k + \frac{1}{k+2}$, then G is $(k, 1)$-decomposable. By using the methodology that is first developed in this paper, a proof of the NDT Conjecture has been obtained by Jiang and Yang in [8]. We expect there will be more applications of these ideas. The presentation of the proof here is more algorithmic, while the proof in [8] was presented by using partially ordered sets.

A slightly different sparseness condition places a bound on the average vertex degree in all subgraphs. The *maximum average degree* of a graph G, denoted by $\text{mad}(G)$, is defined as

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