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The Erdős–Pósa property for edge-disjoint immersions in 4-edge-connected graphs

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ABSTRACT

A graph H is *immersed* in a graph G if the vertices of H are mapped to distinct vertices of G, and the edges of H are mapped to paths joining the corresponding pairs of vertices of G, in such a way that the paths are pairwise edge-disjoint. In this paper, we show that the Erdős–Pósa property holds for packing edge-disjoint K_t-immersions in 4-edge-connected graphs. More precisely, for positive integers k and t, there exists a constant $f(k,t)$ such that a 4-edge-connected graph G has either k edge-disjoint K_t-immersions, or an edge subset F of size at most $f(k,t)$ such that $G-F$ has no K_t-immersion. The 4-edge-connectivity in this statement is best possible in the sense that 3-edge-connected graphs do not have the Erdős–Pósa property.

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1. Introduction

1.1. Erdős–Pósa property

A family $\mathcal{F}$ of graphs is said to have the *Erdős–Pósa property* if for every integer k there is an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint (edge-disjoint, resp.) subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set X of at most $f(k, \mathcal{F})$ vertices (edges, resp.) such that $G - X$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős–Pósa property* arose because Erdős and Pósa [5] prove that the family of cycles has this property. Other families of graphs having the Erdős–Pósa property are the one of cycles of even length [26] and the one of (even) cycles through specified vertices [7,9,16]. Furthermore, there is the seminal result of Robertson and Seymour [18] which says that, for a fixed graph H, the family of H-minors has the Erdős–Pósa property if and only if H is planar.

In this paper, we investigate the Erdős–Pósa property for an edge-disjoint variant of minors for a fixed graph, called *immersions*. For two graphs G and H, an *immersion*[3] of H in G is a map η such that

- $\eta(v) \in V(G)$ for each $v \in V(H)$ and $\eta(u) \neq \eta(v)$ for distinct $u, v \in V(H)$
- for each edge $e = uv$ of H, $\eta(e)$ is a path of G from $\eta(u)$ to $\eta(v)$
- if $e, f \in E(H)$ are distinct, then $\eta(e)$ and $\eta(f)$ have no edges in common, although they may share vertices.

The image of η is called an *H-immersion*.

The immersion relation seems to be similar to the minor relation. In fact, structural approach has been extremely successful for immersions. Robertson and Seymour [23] extended their proof of the famous Wagner's conjecture [22] to prove that graphs are well-quasi-ordered by the immersion relation, which proves a conjecture of Nash-Williams. Recently, the immersion relation has attracted a lot of attention from the graph theory community [2,11,15,27].

Similarly to the minor case, the Erdős–Pósa property does not hold for edge-disjoint immersions of a fixed nonplanar graph. Indeed, consider finding edge-disjoint K_5-immersions. Let $G = (V, E)$ be a graph obtained from a toroidal wall with a large hole with some modification (see Figs. 1 and 2 for example). Then it is not difficult to see that there are no two edge-disjoint K_5-immersions, but we have to delete $\Omega(\sqrt{|V|})$ edges to make G have no K_5-immersion.

As seen in Figs. 1 and 2, the obstruction for K_t-immersions has some topology. Such "topological obstructions" also arise in other examples that do not satisfy the Erdős–Pósa property:

[3] This is often called a *weak immersion*. An immersion η is *strong* if $\eta(e)$ does not contain $\eta(v)$ when e is not incident with v.