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Forbidding induced even cycles in a graph: Typical structure and counting ☆

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We determine, for all $k \geq 6$, the typical structure of graphs that do not contain an induced $2k$-cycle. This verifies a conjecture of Balogh and Butterfield. Surprisingly, the typical structure of such graphs is richer than that encountered in related results. The approach we take also yields an approximate result on the typical structure of graphs without an induced 8-cycle or without an induced 10-cycle.

1. Introduction

1.1. Background

The enumeration and description of the typical structure of graphs with given side constraints has become a successful and popular area at the interface of probabilistic, enumerative, and extremal combinatorics (see e.g. [7] for a survey of such work). For

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example, a by now classical result of Erdős, Kleitman and Rothschild [12] shows that almost all triangle-free graphs are bipartite (given a fixed graph H, a graph is called *H-free* if it does not contain H as a not necessarily induced subgraph). This result was generalised to K_k-free graphs by Kolaitis, Prömel and Rothschild [14]. There are now many precise results on the number and typical structure of H-free graphs and more generally graphs, hypergraphs and other combinatorial structures with a given (anti-)monotone property.

Given a fixed graph H, a graph is called *induced-H-free* if it does not contain H as an induced subgraph. Associated counting and structural questions are equally natural as in the non-induced case, but seem harder to solve. Thus much less is known about the typical structure and number of induced-H-free graphs than that of H-free graphs, though considerable work has been done in this area (see, e.g. [2,4,13,17–19]). In particular, Prömel and Steger [19] obtained an asymptotic counting result for the number of induced-H-free graphs on n vertices, showing that the logarithm of this number is essentially determined by the so-called colouring number of H. This was generalised to arbitrary hereditary properties independently by Alekseev [1] as well as Bollobás and Thomason [8]. Recent exciting developments in [5,20] have opened up the opportunity to replace counting results by more precise results which identify the typical asymptotic structure.

In this paper we determine the typical structure of induced-C_{2k}-free graphs (from which the corresponding asymptotic counting result follows immediately). The key difficulty we encounter is that the typical structure turns out to be more complex than encountered in previous results on forbidden induced subgraphs. This requires new ideas and a more intricate analysis when 'excluding' classes of graphs which might be candidates for typical induced-C_{2k}-free graphs.

1.2. Graphs with forbidden induced cycles

Given a class of graphs $\mathcal{A}$, we let $\mathcal{A}_n$ denote the set of all graphs in $\mathcal{A}$ that have precisely n vertices, and we say that *almost all graphs in $\mathcal{A}$ have property $\mathcal{B}$* if

$$\lim_{n\to\infty} \frac{|\{G \in \mathcal{A}_n : G \text{ has property } \mathcal{B}\}|}{|\mathcal{A}_n|} = 1.$$

Given graphs $H_1, \dots, H_m$, we say *G can be covered by $H_1, \dots, H_m$* if $V(G)$ admits a partition $A_1 \cup \cdots \cup A_m = V(G)$ such that $G[A_i]$ is isomorphic to H_i for every $i \in \{1, \dots, m\}$, where $G[A_i]$ is the subgraph of G induced by A_i.

Prömel and Steger proved in [17] that almost all induced-C_4-free graphs can be covered by a clique and an independent set, and in [16] characterised the structure of almost all induced-C_5-free graphs too. More recently, Balogh and Butterfield [4] determined the typical structure of induced-H-free graphs for a wide class of graphs H. In particular they proved that almost all induced-C_7-free graphs can be covered by either three cliques or two cliques and an independent set, and that for $k \geq 4$ almost all induced-C_{2k+1}-free