# The extremal function for Petersen minors 

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A B S T R A C T

We prove that every graph with $n$ vertices and at least $5 n-8$ edges contains the Petersen graph as a minor, and this bound is best possible. Moreover we characterise all Petersen-minorfree graphs with at least $5 n-11$ edges. It follows that every graph containing no Petersen minor is 9 -colourable and has vertex arboricity at most 5 . These results are also best possible.
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## 1. Introduction

A graph $H$ is a minor of a graph $G$ if a graph isomorphic to $H$ can be obtained from $G$ by the following operations: vertex deletion, edge deletion and edge contraction. The theory of graph minors, initiated in the seminal work of Robertson and Seymour, is at the forefront of research in graph theory. A fundamental question at the intersection of graph minor theory and extremal graph theory asks, for a given graph $H$, what is the maximum number $\mathrm{ex}_{\mathrm{m}}(n, H)$ of edges in an $n$-vertex graph containing no $H$-minor? The function $\operatorname{ex}_{\mathrm{m}}(n, H)$ is called the extremal function for $H$-minors.

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Fig. 1. The Petersen graph.

The extremal function is known for several graphs, including the complete graphs $K_{4}$ and $K_{5}$ [49,10], $K_{6}$ and $K_{7}$ [30], $K_{8}$ [19] and $K_{9}$ [44], the bipartite graphs $K_{3,3}$ [14] and $K_{2, t}$ [6], and the octahedron $K_{2,2,2}$ [8], and the complete graph on eight vertices minus an edge $K_{8}^{-}$[43]. Tight bounds on the extremal function are known for general complete graphs $K_{t}$ [12,23,24,45,46], unbalanced complete bipartite graphs $K_{s, t}$ [25-28], disjoint unions of complete graphs [47], disjoint unions of cycles [15,7], general dense graphs [32] and general sparse graphs $[4,16]$.

### 1.1. Petersen minors

We study the extremal function when the excluded minor is the Petersen graph (see Fig. 1), denoted by $\mathcal{P}$. Our primary result is the following:

Theorem 1. $\operatorname{ex}_{\mathrm{m}}(n, \mathcal{P}) \leqslant 5 n-9$, with equality if and only if $n \equiv 2(\bmod 7)$.
For $n \equiv 2(\bmod 7)$, we in fact completely characterise the extremal graphs (see Theorem 2 below).

The class of $\mathcal{P}$-minor-free graphs is interesting for several reasons. As an extension of the 4 -colour theorem, Tutte [48] conjectured that every bridgeless graph with no $\mathcal{P}$-minor has a nowhere zero 4-flow. Edwards, Robertson, Sanders, Seymour and Thomas [35,37, $36,40,11$ ] have announced a proof that every bridgeless cubic $\mathcal{P}$-minor-free graph is edge 3 -colourable, which is equivalent to Tutte's conjecture in the cubic case. Alspach, Goddyn and Zhang [3] showed that a graph has the circuit cover property if and only if it has no $\mathcal{P}$-minor. It is recognised that determining the structure of $\mathcal{P}$-minor-free graphs is a key open problem in graph minor theory (see [9,31] for example). Theorem 1 is a step in this direction.

### 1.2. Extremal graphs

We now present the lower bound in Theorem 1, and describe the class of extremal graphs. For a graph $H$ and non-negative integer $t$, an ( $H, t)$-cockade is defined as follows: $H$ itself is an $(H, t)$-cockade, and any other graph $G$ is an $(H, t)$-cockade if there are $(H, t)$-cockades $G_{1}$ and $G_{2}$ distinct from $G$ such that $G_{1} \cup G_{2}=G$ and $G_{1} \cap G_{2} \cong K_{t}$. It is well known that for every $(t+1)$-connected graph $H$ and every non-negative integer

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