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Domination in tournaments

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We investigate the following conjecture of Hehui Wu: for every tournament S, the class of S-free tournaments has bounded domination number. We show that the conjecture is false in general, but true when S is 2-colourable (that is, its vertex set can be partitioned into two transitive sets); the latter follows by a direct application of VC-dimension. Our goal is to go beyond this; we give a non-2-colourable tournament S that satisfies the conjecture. The key ingredient here (perhaps more interesting than the result itself) is that we overcome the unboundedness of the VC-dimension by showing that the set of shattered sets is sparse.

1. Introduction

If there is an edge of a digraph G with head v and tail u, we say that "v is adjacent from u" and "u is adjacent to v". If T is a tournament and $X, Y \subseteq V(T)$, we say that X

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dominates Y if every vertex in $Y \setminus X$ is adjacent from some vertex in X. The *domination number* of T is the smallest cardinality of a set that dominates $V(T)$. A class $\mathcal{C}$ of tournaments has *bounded domination* if there exists c such that every tournament in $\mathcal{C}$ has domination number at most c. If S, T are tournaments, we say that T is *S-free* if no subtournament of T is isomorphic to S. A tournament S is a *rebel* if the class of all S-free tournaments has bounded domination. In this paper we investigate the following conjecture, recently proposed by Hehui Wu (private communication):

1.1. Conjecture: *Every tournament is a rebel.*

We will disprove this; and that leads to the question, which tournaments are rebels? We will give a partial answer:

- all 2-colourable tournaments are rebels, and so is at least one more;
- all rebels are poset tournaments.

This needs some definitions. A *k-colouring* of a tournament T is a partition of $V(T)$ into k transitive sets, and if T admits such a partition it is *k-colourable*. The *chromatic number* of a tournament T is the minimum k such that T is k-colourable. We will prove below that all 2-colourable tournaments are rebels, using VC-dimension; and since not all tournaments are rebels, one might anticipate the converse, that all rebels are 2-colourable. The main goal of this paper is to give a counterexample to this. The tournament on seven vertices, obtained by substituting a cyclic triangle for two of the three vertices of a cyclic triangle, is not 2-colourable, but we will show it is a rebel. This is proved in sections 5 and 6. Again the proof uses VC-dimension, using an extension of a theorem of Haussler and Welzl [5], proved in section 4, that permits large shattered sets provided they are sparse.

Let us say a tournament is a *poset* tournament if its vertex set can be ordered $\{v_1, \dots, v_n\}$ such that for all $i < j < k$, if v_j is adjacent from v_i and adjacent to v_k then v_i is adjacent to v_k; that is, the "forward" edges under this linear order form the comparability graph of a partial order. In section 2 we prove that every rebel is a poset tournament, and consequently disprove 1.1.

Domination in tournaments is an old and much-studied question [6]. For instance, let us say a tournament T is *k-majority* if there are $2k-1$ linear orders on $V(T)$ such that for all distinct $u, v \in V(T)$, if u is adjacent to v then u is before v in at least k of the $2k-1$ orders. Alon, Brightwell, Kierstead, Kostochka and Winkler showed in [1] that k-majority tournaments have bounded domination number, and indeed this paper is where the idea of using VC-dimension for tournament domination was introduced. Their result follows from the fact that 2-colourable tournaments are rebels, since it is easy to see (by estimating the number of n-vertex tournaments in each class) that some 2-colourable tournament S is not k-majority; and since S is a rebel and the class of k-majority tournaments is S-free, the latter has bounded dom-