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On the decomposition of random hypergraphs

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A R T I C L E I N F O

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A B S T R A C T

For an r-uniform hypergraph H, let $f(H)$ be the minimum number of complete r-partite r-uniform subhypergraphs of H whose edge sets partition the edge set of H. For a graph G, $f(G)$ is the bipartition number of G which was introduced by Graham and Pollak in 1971. In 1988, Erdős conjectured that if $G \in G(n, 1/2)$, then with high probability $f(G) = n - \alpha(G)$, where $\alpha(G)$ is the independence number of G. This conjecture and its related problems have received a lot of attention recently. In this paper, we study the value of $f(H)$ for a typical r-uniform hypergraph H. More precisely, we prove that if $(\log n)^{2.001}/n \leq p \leq 1/2$ and $H \in H^{(r)}(n, p)$, then with high probability $f(H) = (1 - \pi(K_r^{(r-1)}) + o(1))\binom{n}{r-1}$, where $\pi(K_r^{(r-1)})$ is the Turán density of $K_r^{(r-1)}$.

1. Introduction

For a graph G, the *bipartition number* $\tau(G)$ is the minimum number of complete bipartite subgraphs of G so that each edge of G belongs to exactly one of them. This parameter of a graph was introduced by Graham and Pollak [12] in 1971. The famous Graham–Pollak [12] Theorem asserts $\tau(K_n) = n - 1$. Since its original proof using

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Sylvester's Law of Inertia, many other proofs have been discovered, see [16], [17], [18], [19], [20], [21].

Let $\alpha(G)$ be the independence number of G. It is easy to observe $\tau(G) \leq |V(G)| - \alpha(G)$. Erdős (see [13]) conjectured that the equality holds for almost all graphs. Namely, if $G \in G(n, 1/2)$, then $\tau(G) = n - \alpha(G)$ with high probability. Alon [2] disproved this conjecture by showing $\tau(G) \leq n - \alpha(G) - 1$ with high probability for most values of n. Improving Alon's result, Alon, Bohman, and Huang [3] proved that if $G \in G(n, 1/2)$, then with high probability $\tau(G) \leq n - (1 + c)\alpha(G)$ for some positive constant c. Chung and the author [6] proved that if $G \in G(n, p)$, p is a constant, and $p \leq 1/2$, then with high probability we have $\tau(G) \geq n - \delta(\log_{1/p} n)^{3+\epsilon}$ for any constants δ and ϵ. When p satisfies $\frac{2}{n} \leq p \leq c$ for some absolute (small) constant c, Alon [2] showed that if $G \in G(n, p)$, then $\tau(G) = n - \Theta\left(\frac{\log(np)}{p}\right)$ with high probability.

The hypergraph analogue of the bipartition number is well-defined. For $r \geq 3$ and an r-uniform hypergraph H, let $f(H)$ be the minimum number of complete r-partite r-uniform subhypergraphs of H whose edge sets partition the edge set of H. Aharoni and Linial (see [1]) first asked to determine the value of $f(K_n^{(r)})$ for $r \geq 3$, where $K_n^{(r)}$ is the complete r-uniform hypergraph with n vertices. The value of $f(K_n^{(r)})$ is related to a perfect hashing problem from computer science. Alon [1] proved $f(K_n^{(3)}) = n - 2$ and $c_1(r)n^{\lfloor \frac{r}{2} \rfloor} \leq f(K_n^{(r)}) \leq c_2(r)n^{\lfloor \frac{r}{2} \rfloor}$ for $r \geq 4$. For improvements and variations, readers are referred to [7], [8], [9], [10], [14], and [15]. For each real $0 \leq p \leq 1$, let $H^{(r)}(n, p)$ denote the random r-uniform hypergraph in which each r-set $F \in \binom{[n]}{r}$ is selected as an edge with probability p independently. In this paper, we examine the value of $f(H)$ for the random hypergraph $H \in H^{(r)}(n, p)$. To state our main theorem, we need a few more definitions.

For an r-uniform hypergraph H, the *Turán number* $\mathrm{ex}(n, H)$ is the maximum number of edges in an n-vertex r-uniform hypergraph which does not contain H as a subhypergraph. We define the *Turán density* of H as

$$\pi(H) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{r}}.$$

For each $r \geq 3$, we use $K_r^{(r-1)}$ to denote the compete $(r-1)$-uniform hypergraph with r vertices.

By extending techniques from [2] and [6], we are able to prove the following theorem.

Theorem 1. *For $r \geq 3$, if $(\log n)^{2.001}/n \leq p \leq 1/2$ and $H \in H^{(r)}(n, p)$, then with high probability we have*

$$f(H) = (1 - \pi(K_r^{(r-1)}) + o(1))\binom{n}{r-1}.$$

From this theorem, we can see the typical value of $f(H)$ has the order of magnitude n^{r-1} while $f(K_n^{(r)})$ has the order of magnitude $n^{\lfloor \frac{r}{2} \rfloor}$. We note $\pi(K_3^{(2)}) = \frac{1}{2}$ while the

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